

NASA Contractor Report 3817

Theory and Modeling of Atmospheric Turbulence

Part II: September 1, 1982–August 31, 1983

C. M. Chen

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***The City College Research Foundation
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FOREWORD

The domain of statistical physics has extended from the traditional problems in quantum mechanics, the solid states and the kinetic theory of gases to other nonlinear problems in hydrodynamics, plasmas and optical systems, for which we need to consider the fluctuation effects. Conceptually, it has been found very helpful to view these statistical problems from a basic physical point of view which emphasizes their structural similarity.

The study of stochastic systems has developed in two directions. First, the transition from a laminar state to a chaotic or turbulent state through a sequence of bifurcations shows a certain universality (see "Chaos and Universality," Nordita Selection, 1981). Secondly, when the system has reached a state of fully developed chaos or strong turbulence, the statistical methods used for their theoretical analysis find again a certain universality. A proper understanding of the many basic phenomena in astrophysics, space, atmospherical and optical applications depends critically on our ability to analyze the turbulent characteristics and the collective processes in these nonlinear systems. The lack of a suitable methods of treatment and the difficulties encountered even for the simplest form of incompressible, homogeneous and isotropic turbulence, have hindered the theoretical development of strong turbulence.

From the physical point of view, a turbulent state is characterized by its transport properties: eddy diffusivity, eddy viscosity, coefficient of damping, or amplification. Their analytical determination requires a transport theory. The kinetic method is best suited for this purpose. The eddy transport coefficients, as induced by the fluctuations of small scales, govern the evolution of larger scales. This requirement of scaling leads to the concept of renormalization groups, from which we develop the group-kinetic method of turbulence. We have applied the method to problems of atmospheric turbulence.

The group-kinetic method combines the advantages of the kinetic method and the group-scaling. The kinetic description has the advantage of transforming the system of hydrodynamical equations that govern a nonlinear stochastic system into a master equation of lesser nonlinearity. The group-scaling enables the determination of the transport properties and the spectral structure by the one-point distribution function alone, without involving the two-point distribution. Our closure is obtained by a memory loss in the relaxation process and not by truncation of the infinite hierarchy of n -point distribution functions or correlation functions.

The six verbatim sections include research accomplished during the second year, September 1, 1982 through August 31, 1983, of this two-year contractual effort.

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SECTION 1

General Considerations on the Group-Kinetic Theory of Turbulence

■ - PRINCIPLE OF THE GROUP-KINETIC THEORY

The Navier-Stokes equation of motion with external and self-consistent forces is transformed into a master equation

$$(\partial_t + \hat{L})\hat{f}(t, \underline{x}, \underline{v}) = 0,$$

for the distribution function $\hat{f}(t, \underline{x}, \underline{v})$ of velocity \underline{v} , with the equivalence relation

$$f(t, \underline{x}, \underline{v}) = \rho \delta[\underline{v} - \hat{u}(t, \underline{x})],$$

that is valid for a constant or variable density ρ , and for a fluctuating fluid velocity $\hat{u}(t, \underline{x})$. Here \hat{L} is the differential operator for the perturbation of the trajectory in the phase space.

The total distribution function

$$\hat{f} = \bar{f} + \tilde{f}$$

consists of a mean distribution $\bar{f} = \langle \hat{f} \rangle$ and a fluctuation \tilde{f} . The usual Fourier decomposition of a fluctuating function contains too many minute details, and a coarse-graining procedure is necessary. In analogy with the "renormalization groups", we decompose \tilde{f} into groups

$$f^0, f', f''$$

of decreasing coherence, representative of the spectral evolution, eddy diffusivity, and relaxation, respectively. By formulating the relaxation as a functional of the diffusivity, we obtain a closure.

The group-scaling has the advantage of determining the spectrum from the

singlet distribution function alone, without involving the 2-point distribution function as is required in the conventional methods of statistical mechanics.

The kinetic equation is derived for f^0 . It takes the generalized form of the Fokker-Planck equation, with the diffusivity $\mathcal{D}\{ \}$ as an integral operator to represent the memory and the collective behavior. It is converted into the hydrodynamic equation in the group form, from which we determine the eddy viscosity

$$\kappa' \left[F(k), \text{parameters} \right]$$

and calculate the spectral function $F(k)$. The parameters include the Coriolis field and the length of stability from shear and buoyancy.

II. SPECTRAL DISTRIBUTION OF TURBULENCE

A. The application of the group-kinetic method to the Navier-Stokes equation of motion finds the direct and reverse cascades for the transfer across a spectrum, and derives the spectral distributions in inertia ($k^{-5/3}$), shear (k^{-1}), and geostrophic turbulence (k^{-3} and k^{-4}). In boundary layer turbulence, the spectra of velocity fluctuations in separate directions can be treated, and a spectral gap in the stable boundary layer is expected.

B. The extensions to the Zakharov equations, the nonlinear Schrödinger equation and the Korteweg-deVries equation are important for analyzing the new dynamical characteristics and transport properties of soliton turbulence in fluids, plasmas and solid, and the self-focusing in optical turbulence.

III. NUMERICAL MODELING

The kinetic equation for \bar{f} and its conversion into fluid representation forms the basis for the modeling and prediction of turbulent profiles and

spectra in the turbulent boundary layer. Here the transport coefficients become explicit functions of t , x and the governing parameters, once the spectral laws are analytically determined by our group-kinetic theory. The **pressure-velocity** correlation is also analytically derived.

Our first aim in the modeling of turbulence is to provide an analytical basis for the Monin-Obukhoff similarity by determining the structure of the universal functions. Then, we can devise a new numerical **modeling** which will incorporate the effects of scales, the collective behavior and the memory.

IV. ATMOSPHERIC BOUNDARY LAYER

We can divide the turbulent atmospheric boundary layer into:

- A. The lower boundary layer, or surface layer
- B. The upper boundary layer which includes the free-convection layer, the mixed layer, and the Ekman layer.

The surface layer extends from the ground up to a height of less than 100 meters, and is characterized by the constant fluxes of momentum, temperature and humidity. The main tasks are the prediction of profiles and spectra by analytically determining the universal **functions that could not be determined by the empirical similarity theory of Monin and Obukhoff.**

The upper layer contains the additional effect of the Coriolis force, which enters into the differential operator in perturbing the trajectory.

The above division of the atmospheric turbulence into two parts, as dependent on the absence **or** presence of the Coriolis field, is analogous to the classification of plasma turbulence into two parts: the **Langmuir** turbulence (**i.e.** without magnetic field) and the magnetized plasma turbulence.

SECTION 2

Group-Kinetic Theory of Two-Dimensional Geostrophic Turbulence

C. M. Tchen

The City College and The Graduate Center
of The City University of New York, N. Y. 10031

ABSTRACT

The two-dimensional geostrophic turbulence driven by a random force is investigated. On the basis of the Liouville equation which simulates the primitive hydrodynamical equations, we develop a group-kinetic theory of turbulence and derive the kinetic equation of the scaled singlet distribution. This distribution will suffice for the investigation of the spectrum of turbulence, without having to resort to the pair-distribution as was with the usual kinetic theories. The collision integral has a memory and describes the pair interaction and its enhancement by the multiple interaction.

Our kinetic equation of turbulence is transformed into an equation of spectral balance in the equilibrium and non-equilibrium states, as governed by the direct cascade and the reverse cascade, respectively. The sequence of power laws k^{-3} , k^{-4} of velocity fluctuations are derived.

I. INTRODUCTION

The two-dimensional geostrophic turbulence has attracted much attention [Fjørtoft, 1953; Charney, 1971; Rhines, 1979] . Theories based upon the diffusion approximation [Leith, 1968], the spectral properties [Novikov, 1979], the direct interaction approximation [Kraichnan, 1967, 1971a,b ; Pouquet et al., 1975; Salmon et al., 1978], and numerical modeling and simulations [Lilly, 1969a,b, Basdevant et al., 1978] have found a spectral k^{-3} . This spectrum changes into k^{-4} if the geostrophic turbulence is driven by a random force [Saffman, 1971; Thompson, 1973] . Deviations from these laws have also been discussed [Gage, 1979] . The sequence of appearance of the spectra k^{-3} , k^{-4} , and the dynamics of the direct and reverse cascade were analytically vague in the lack of a generalized statistical theory of the geostrophic turbulence driven by a random force.

Our investigation is divided into two parts. The first part develops a group-kinetic theory on the basis of our earlier scaling procedure [Tchen, 1978, 1979] , and derives the kinetic equation of turbulence. Usually the analysis of the spectral structure requires either a pair-distribution function or a detailed Fourier decomposition. These are not needed here, because of the scaling procedure mentioned. Our kinetic approach has the advantage of not only including the random force in our homogeneous Liouville equation, but also of describing the interaction between the wave and the fluid particle, as characteristic of the large scale turbulence. The collective collision obtained will include the memory, the pair-collision and the multiple collision [Tchen and Misguich, 1982] . The latter effect is essential and will be applied to the treatment of the reverse cascade that is often found with the large scale turbulence.

With the kinetic foundation described above, the kinetic equation of turbulence is transformed into its hydrodynamical form, and is used to develop the spectral theory of geostrophic turbulence in the second part of this paper. The **two** parts are separately self-consistent and can be read independently.

11. MICRODYNAMICAL STATE OF TURBULENCE

The Navier-Stokes equation of motion

$$(\partial_t + \hat{\underline{u}} \cdot \nabla - \nu \nabla^2) \hat{\underline{u}} = \hat{\underline{E}} \quad (1)$$

of an incompressible fluid satisfies the condition

$$\nabla \cdot \hat{\underline{u}} = 0. \quad (2)$$

By taking the curl, we obtain the vorticity

$$\hat{\underline{\zeta}} = \nabla \times \hat{\underline{u}} \quad (3)$$

and transform (1) into the following vorticity equation of the two-dimensional geostrophic turbulence:

$$(\partial_t + \hat{\underline{u}} \cdot \nabla - \nu \nabla^2) \hat{\underline{\zeta}} = \nabla \times \hat{\underline{E}}. \quad (4)$$

Here we have

$$\hat{\underline{u}} = (u_1, u_2, 0) \quad \text{and} \quad \hat{\underline{\zeta}} = (0, 0, \hat{\zeta}). \quad (5)$$

The field

$$\hat{\underline{E}} = \hat{\underline{E}}_P + \hat{\underline{E}}_X \quad (6)$$

has **two** components. The component

$$\hat{\underline{E}}_P = -\frac{1}{\rho} \nabla \hat{P}, \quad (7)$$

which is the gradient of pressure \hat{P} at a constant density ρ , is not present in the vorticity equation (3), because

$$\nabla \times \hat{\underline{E}}_P = -\frac{1}{\rho} \nabla \times \nabla \hat{P} = 0. \quad (8)$$

The other component $\hat{\underline{\underline{E}}}_{\underline{\underline{x}}}$ forms a finite vorticity source $\underline{\underline{\nabla}} \times \hat{\underline{\underline{E}}}_{\underline{\underline{x}}}$.

The equations (1), (2) and (4) describe the microdynamical state of turbulence, and are used as the point of departure of our statistical treatment. In our transport theory by eddies larger than the viscous cutoff, the effect of the kinematic viscosity will be neglected, but will be restituted in the spectral balance where the viscosity gives a molecular dissipation.

For the development of a kinetic method, we write the microkinetic equation in the form of the Liouville equation:

$$\left[\partial_t + \hat{\underline{\underline{L}}}(\underline{\underline{t}}) \right] \hat{\underline{\underline{f}}}(\underline{\underline{t}}, \underline{\underline{x}}, \underline{\underline{v}}) = 0, \quad (9)$$

with the differential operator

$$\hat{\underline{\underline{L}}}(\underline{\underline{t}}) = \underline{\underline{v}} \cdot \underline{\underline{\nabla}} + \hat{\underline{\underline{E}}}(\underline{\underline{t}}, \underline{\underline{x}}) \cdot \underline{\underline{\partial}} - \nu \nabla^2, \quad \underline{\underline{\partial}} \equiv \partial / \partial \underline{\underline{v}}. \quad (10)$$

The detailed distribution function $\hat{\underline{\underline{f}}}$ is normalized to unity, as

$$\int d\underline{\underline{v}} \hat{\underline{\underline{f}}}(\underline{\underline{t}}, \underline{\underline{x}}, \underline{\underline{v}}) = 1, \quad (11)$$

and assumes the form

$$\hat{\underline{\underline{f}}}(\underline{\underline{t}}, \underline{\underline{x}}, \underline{\underline{v}}) = \delta \left[\underline{\underline{v}} - \hat{\underline{\underline{u}}}(\underline{\underline{t}}, \underline{\underline{x}}) \right], \quad (12)$$

in order to be consistent with the Navier-Stokes equation (1) and the equation of continuity (2). It is not difficult to show by taking the first two moments, that the Liouville equation will reproduce the two hydrodynamical equations (1) and (2). The Liouville equation is homogeneous and has less nonlinear terms than does the Navier-Stokes equation, because the function $\hat{\underline{\underline{u}}}(\underline{\underline{t}}, \underline{\underline{x}})$ of velocity of fluid is replaced by an independent variable $\underline{\underline{v}}$. We retain the only nonlinear term $\hat{\underline{\underline{E}}} \cdot \underline{\underline{\partial}} \hat{\underline{\underline{f}}}$ in order to describe the mode-couplings.

III. GROUP-SCALING PROCEDURE

The Liouville equation (9) contains too many minute details which are unnecessary for a statistical treatment of turbulence. We apply a scaling procedure by dividing the detailed distribution

$$\hat{f} = \bar{f} + \tilde{f} \quad (13)$$

into a mean distribution \bar{f} and a fluctuation \tilde{f} , and the fluctuation

$$\tilde{f} = f^{\circ} + f' \quad (14)$$

into a macro-group f° and a micro-group f' . The function $\bar{f}(t, \underline{x}, \underline{v})$ gives the distribution of velocity in non-equilibrium. The macro-group f° calculates the spectrum of turbulence without the need of developing a separate kinetic equation of the pair-distribution function $\langle \tilde{f} \tilde{f} \rangle$. In the following, we develop a kinetic equation of the macro-group f° and investigate the spectrum of turbulence.

The evolution of the macro-group is controlled by the transport property as shaped by the fluctuations of the micro-group, while the approach of the transport property to equilibrium is obtained by the relaxation. These three transport processes of evolution, transport property and relaxation are represented by the three groups

$$f^{\circ}, f', f''$$

of the distribution function, or by the three groups

$$E^{\circ}, E', E''$$

of the field. The interactions between the groups are described by a coupled system of equations.

The groups have their time scales characterized by the correlation times

$$\tau_c^{\circ} > \tau_c' > \tau_c'' \quad (15a)$$

For formulating the transport coefficients, the correlations, as given by the groups $E'(t-\tau)$, $E''(\tau-\tau')$, are integrated with respect to τ and τ' in the domains $(0, t)$, $(0, \tau)$, respectively, so that the evolution times t , τ , τ' can be ranged among the correlation times (15a), in the following manner:

$$t > \tau_c' > \tau > \tau_c'' > \tau'. \quad (15b)$$

The degradation of coherence (15) will constitute a property of quasi-stationarity of one group with respect to the other.

The individual fluctuating groups

$$u^0, u', u''$$

can be decomposed into Fourier components with overlapping wavenumbers. But the global averages, a g,

$$\langle (u^0)^2 \rangle, \langle (u')^2 \rangle, \langle (u'')^2 \rangle$$

are deterministic, and are separated by their adjacent wavenumber domains

$$(0, k), \quad (k, \infty), \quad (k' \geq k, \infty) \quad (16)$$

with k, k' varying from 0 to ∞ .

For the sake of convenience, we introduce the scaling operators

$$\bar{A}, A^0, A', A'' \quad (17a)$$

$$A_0 = \bar{A} + A^0, \quad \tilde{A} = 1 - \bar{A}, \quad A^{(1)} = A' - A'' \quad (17b)$$

IV. SCALING OF THE LIOUVILLE EQUATIONS

By applying the operators A^0 and A' to scale the Liouville equation (9), we obtain the system of coupled equations:

$$(\partial_t + A^0 \hat{L}) f^0 = -L^0 \bar{f} + c^0, \text{ with } c^0 \equiv -A^0 L' f' \quad (18)$$

$$(\partial_t + \hat{L}) f' = -L' f_0 - c_0. \quad (19)$$

The second equation of the system may be equivalently replaced by

$$(\partial_t + A' \hat{L}) f' = -L' f_0. \quad (20)$$

Upon integrating (19) and (20), we have

$$\begin{aligned} f' &= -A' \int_0^t d\tau \hat{U}(t, t-\tau) L'(t-\tau) f_0(t-\tau) \\ &\quad - A' \int_0^t d\tau \hat{U}(t, t-\tau) c_0(t-\tau), \end{aligned} \quad (21)$$

and

$$f' = -A' \int_0^t d\tau \hat{\Lambda}(t, t-\tau) L'(t-\tau) f_0(t-\tau) \quad (22)$$

Here \hat{U} and $\hat{\Lambda}$ are evolution operators as related to the differential operators \hat{L} and $A' \hat{L}$, respectively. The initial value $f'(0)$ has been neglected, since it cannot produce a finite correlation $A^0 L'(t) f'(0)$ at large t , by (15).

Subsequently, we premultiply (21) and (22) by $-A^0 L'$ to find the collision in the following two forms:

$$\begin{aligned} c^0 &= c^0 + A^0 \int_0^t d\tau L'(t) U'(t, t-\tau) c_0(t-\tau) \\ &= c^0 - A^0 H^0 * c_0, \text{ with } H^0 \equiv -A^0 L'(t) U'(t, t-\tau) \end{aligned} \quad (23)$$

$$C^0 = A^0 \Delta' \left\{ f^0(t-\tau) \right\} . \quad (24)$$

The collective collision C^0 is seen to consist of two parts: The pair collision

$$C^0 = A^0 \Delta' \left\{ f^0(t-\tau) \right\} , \quad (25)$$

from the correlation between two micro-fields $A^0 L' A' \hat{U} L'$, is proportional to Δ' . The multiple collision, from the correlation $A^0 L' U' C_0$ between a micro-field and the cluster of the organized micro-fluctuations, is proportional to Δ'^2 .

The governing collision operators

$$\begin{aligned} A^0 \Delta' &= A^0 \int_0^t d\tau L'(t) A' \hat{U}(t, t-\tau) L'(t-\tau) \\ &= \partial \cdot A^0 \underline{D}' \cdot \partial \end{aligned} \quad (26a)$$

$$\begin{aligned} A^0 \Delta' &= A^0 \int_0^t d\tau L'(t) A' \hat{\Lambda}(t, t-\tau) L'(t-\tau) \\ &= \partial \cdot A^0 \underline{D}' \cdot \partial \end{aligned} \quad (26b)$$

are defined by the diffusivities

$$A^0 \underline{D}' = A^0 \int_0^t d\tau E'(t) A' \hat{U}(t, t-\tau) E'(t-\tau) \quad (27a)$$

$$A^0 \underline{D}' = A^0 \int_0^t d\tau E'(t) A' \hat{\Lambda}(t, t-\tau) E'(t-\tau) \quad (27b)$$

which themselves may serve as operators. In (23) we have replaced

$$A^0 L'(t) A' \hat{U}(t, t-\tau) C_0(t-\tau) \quad \text{by} \quad A^0 L'(t) U'(t, t-\tau) C_0(t-\tau)$$

without loss of generality.

The operator $\hat{U}(t, t-\tau)$, with $t' = t-\tau$, is governed by the Liouville equations:

$$[\partial_t + \hat{L}(t)] \hat{U}(t, t') = 0 \quad (28a)$$

$$[\partial_{t'} - \hat{L}(t')] \hat{U}(t, t') = 0. \quad (28b)$$

The operator \hat{A} is related to \hat{U} , and the relation has been developed by Weinstock [1969].

By scaling (28a) by means of A'' , we obtain the kinetic equation

$$(\partial_t + L_0)U^0 = -L^0 \bar{U} + H^0, \quad (29)$$

with the collisions

$$H^0(t, t') = A^0 \Delta' \{U^0(t, t')\} \quad (30a)$$

$$H^0(t, t') = A^0 \Delta' \{U^0(t, t')\}, \quad (30b)$$

as related by

$$H^0 = H^0 - A^0 H^0 \times H^0, \quad (30c)$$

in analogy with (23) - (25).

V. PROPAGATOR AND PROBABILITY OF RETROGRADE TRANSITION

The Liouville equations (9) and (28) which govern \hat{f} and \hat{U} have the same differential operator \hat{L} , and therefore the same characteristic equations governing the dynamical variables in the form

$$\frac{d\hat{x}(t')}{dt'} = \hat{v}(t') \quad , \quad \frac{d\hat{v}(t')}{dt'} = \hat{E}(t') \quad , \quad (31a)$$

or equivalently,

$$\frac{d^2\hat{x}(t')}{dt'^2} = \hat{E}(t') \quad , \quad (31b)$$

with the conditions

$$\hat{x}(t) = \underline{x}, \quad \hat{v}(t) = \underline{v}, \quad (31c)$$

by the nature of the differential equation of the second degree, where \underline{x} and \underline{v} are two independent variables. The values (31c) will be called the "initial conditions" in the retrograde transition from t to t' .

By a change of variables

$$\hat{x}(t-\tau) = \underline{x} - \hat{\ell}(\tau) \quad , \quad \hat{v}(t-\tau) = \underline{v} - \hat{v}(\tau), \quad \tau = t - t', \quad (32a)$$

we can write the characteristic equations in the integral form

$$\hat{\ell}(\tau) = \int_0^\tau d\tau' \hat{v}(t-\tau') \quad (32b)$$

$$\hat{v}(\tau) = \int_0^\tau d\tau' \hat{E}(t-\tau') \quad . \quad (32c)$$

Note that $\hat{x}(t-\tau)$ from (32a) can be rewritten as

$$\hat{x}(t-\tau) = \underline{x} \left[t-\tau, \underline{x} - \hat{\ell}(\tau | t, \underline{x}) \right]$$

or as

$$\hat{\underline{x}}(t-\tau) \equiv \hat{\underline{x}} \left[t-\tau, \underline{x}-\hat{\underline{l}}(\tau) \right]$$

in the Lagrangian representation, with

$$\hat{\underline{l}}(\tau | t, \underline{x}) \text{ replaced by } \hat{\underline{l}}(\tau)$$

for the sake of brevity. The vertical bar in $|_{t, \underline{x}}$ denotes that the Lagrangian displacement $\hat{\underline{l}}(\tau)$ was made during a time interval τ along the trajectory that passes by the point \underline{x} at the time instant t .

The systems (31) and (32) are called the equations of the path dynamics in the differential form and the integral form, respectively. Both forms equally describe the trajectory of a fluid particle which occupies the position $\hat{\underline{x}}(t')$ at the time instant t' , provided the trajectory passes by the point \underline{x} at the instant t while having a velocity \underline{v} .

It is to be stipulated that the essential function of the propagator \hat{U} is to impose a Lagrangian representation of the function, say \hat{E} as in (27a), in the form

$$\hat{E} \left[t', \hat{\underline{x}}(t') \right] \equiv \hat{U}(t, t') \hat{E}(t'), \quad (33a)$$

following the trajectory that is determined by the differential path dynamics (31).

Alternatively, with the path dynamics (32), the Lagrangian representation can be obtained in the integral form

$$\hat{E} \left[t-\tau, \underline{x}-\hat{\underline{l}}(\tau) \right] = \int d\underline{l} \hat{E}(t-\tau, \underline{x}-\underline{l}) \hat{p}(t-\tau, \underline{x}-\underline{l} | t, \underline{x}), \quad (33b)$$

by using the probability density

$$\hat{p}(t-\tau, \underline{x}-\underline{l} | t, \underline{x}) . \quad (34)$$

The integral relation is written for the purpose of transforming the Lagrangian function into the Eulerian function.

The probability density satisfies the condition of normalization

$$\int d\vec{l} \hat{p} = 1. \quad (35)$$

The density (34a) defines the probability

$$\hat{p}(t-\tau, \vec{x}-\vec{l} \mid t, \vec{x}) d\vec{l}$$

for the fluid particle to occupies a position between

$$\vec{x}-\vec{l} \quad \text{and} \quad \vec{x}-(\vec{l}+d\vec{l})$$

at the time instant $t-\tau$, provided it follows the trajectory which passes by the point \vec{x} at the time t . For the retrograde transition to be specified by the integral path dynamics of $\hat{\vec{l}}(\tau)$, as given by (32), we write [Tchen, 1944]

$$\hat{p}(t-\tau, \vec{x}-\vec{l} \mid t, \vec{x}) = \delta[\vec{l} - \hat{\vec{l}}(t-\tau, \mid t, \vec{x})],$$

or briefly

$$\hat{p}(\tau, \vec{l}) = \delta[\vec{l} - \hat{\vec{l}}(\tau)] . \quad (36)$$

The abbreviated form indicates a quasi-stationarity of \hat{p} in the t, \vec{x} space, so that the Liouville equation can be written in the form:

$$[\partial_\tau + \hat{\mathcal{L}}(t-\tau)] \hat{p}(t-\tau, \vec{x}-\vec{l} \mid t, \vec{x}) = 0, \quad (37a)$$

with the differential operator

$$\hat{\mathcal{L}}(t-\tau) = -\hat{\vec{v}}(t-\tau) \cdot \nabla_{\vec{l}}, \quad \nabla_{\vec{l}} = \partial/\partial \vec{l}. \quad (37b)$$

It is not difficult to show by moments that (37a), together with (36), will reproduce the integral path dynamics with the initial velocity \vec{v} that is a random variable having a distribution $\hat{f}(t, \vec{x}, \vec{v})$. By definition (12), \vec{v} can be identified as $\hat{\vec{u}}(t, \vec{x})$ in the return to the

micro-hydrodynamical description. Thus the Lagrangian representation (33a) by means of the propagator $\hat{U}(t, t')$ is straight forwardly kinetic. On the other hand, the Lagrangian representation (33b) by means of the probability $\hat{p}(\tau, \underline{\ell})$ is valid in both the fluid and kinetic descriptions. In the former, the initial velocity is $\hat{u}(t, \underline{x})$, and the Liouville equation (37a) becomes dissociated from the distribution $\hat{f}(t, \underline{x}, \underline{v})$. In the latter, although the initial velocity \underline{v} is not directly involved as a variable in the probability density $\hat{p}(\tau, \underline{\ell})$, it is implicit in the path dynamics through the distribution $\hat{f}(t, \underline{x}, \underline{v})$, so that the same Liouville equation can be considered as an equation of a kinetic significance in the contracted dimensionality, i.e. in the sense that the governing path dynamics requires an initial velocity to be determined kinetically. The contracted dimensionality gives to the Liouville equation a simpler differential operator that is independent of $\underline{\ell}$ and therefore a simpler Fourier transformation.

In the following, we shall exploit this advantage of the contracted dimensionality by using the probability of transition in transforming a Lagrangian function, e.g. (33a), into an Eulerian function, e.g. (33b).

VI. MEMORY IN THE COLLECTIVE COLLISION

The problem of the derivation of the collective collision c^0 in terms of the pair collision C^0 involves the solution of the coupled integral equations (23) and (30), and the treatment of the Lagrangian functions associated with the propagator and its collision, from (29) and (30b). The problem is complicated. We shall make the simplification by separating the chain of correlations (15b) into two sections

$$t > \tau_c' > \tau \quad (38a)$$

and

$$\tau > \tau_c' > \tau' \quad (38b)$$

referring to the processes of transport coefficient, i.e. diffusivity, and relaxation, i.e. propagator, respectively. The first section which **has** a correlation of long duration shall preserve **its** memory, while the correlation of the second section which has a short duration should not propagate its memory beyond the time span τ and exert an effect on the evolution of $f^0(t, \underline{x}, \underline{v})$,

With this memory-loss we simplify the equations of collisions (23) and (30b) into the system of equations

$$c^0 = C^0 - \bar{H} * c^0 \quad (39a)$$

$$\bar{H} = \langle \Delta' \rangle \bar{U}, \quad (39b)$$

which we can combine into

$$c^0 = C^0 - \langle \Delta' \rangle \bar{U} * c^0. \quad (40)$$

It leaves the Lagrangian-Eulerian transformation by the method of the transition probability, as follows:

$$\begin{aligned} \bar{H} * c^0 &= \langle \Delta' \rangle \bar{U} * c^0 \\ &= \langle \Delta' \rangle \int_0^t d\tau \int d\underline{l} \bar{p}(\tau, \underline{l}) c^0(t-\tau, \underline{x}-\underline{l}). \end{aligned} \quad (41)$$

We see the transformation

$$\bar{U}(t, t-\tau) C^0(t-\tau) = \int d\underline{\ell} \bar{P}(\tau, \underline{\ell}) C^0(t-\tau, \underline{x}-\underline{\ell})$$

from (33b), in which the kinetics of \bar{U} as governed by (29) is replaced by the dynamics of \bar{P} as governed by (37a). Note that

$$C^0, \bar{H}, \langle \Delta' \rangle, \bar{U}$$

are in the $t, \underline{x}, \underline{v}$ space. On the other hand, in the $\tau, \underline{\ell}$ space, $\bar{P}(\tau, \underline{\ell})$ does not carry \underline{v} , except in the path dynamics as an initial condition.

Further simplification is obtained by a Fourier transformation of (41) into

$$\bar{H} * C^0 = \langle \Delta' \rangle (2\pi)^{-d} \pi \bar{P}(\omega, \underline{k}) C^0(\omega, \underline{k}) = -\mathcal{K}(\omega, \underline{k}) C^0(\omega, \underline{k}), \quad (42a)$$

with

$$\mathcal{K}(\omega, \underline{k}) = -\langle \Delta' \rangle \pi \int_0^\infty dt e^{i(\omega - \underline{k} \cdot \underline{x})t} \bar{P}(t, \underline{k}) \quad (42b)$$

in d-dimensions, so that the integral equation (40) is transformed into the form

$$C^0(\omega, \underline{k}) = C^0(\omega, \underline{k}) + \mathcal{K}(\omega, \underline{k}) C^0(\omega, \underline{k}),$$

or, by collecting C^0 ,

$$C^0(\omega, \underline{k}) = \beta(\omega, \underline{k}) C^0(\omega, \underline{k}), \quad (43)$$

with

$$\beta(\omega, \underline{k}) = |1 - \mathcal{K}(\omega, \underline{k})|^{-1}. \quad (44)$$

The probability function $\bar{P}(\tau, \underline{\ell})$ is determined by the following differential equation of evolution:

$$\partial_\tau \bar{P} = \nabla_{\underline{\ell}} \cdot \langle \tilde{K}(\tau) \rangle \cdot \nabla_{\underline{\ell}} \bar{P}, \quad (45)$$

that is scaled from the Liouville equation

$$(\partial_\tau + \tilde{\mathcal{L}}) \hat{P} = 0 \quad (46)$$

with a fluctuating differential operator. We find the solution:

$$\bar{P}(\tau, k) = \exp \left[-k^2 \int_0^\tau d\tau' \langle \tilde{K}(\tau') \rangle \right] , \quad (47)$$

From the path dynamics (32), we find

$$\begin{aligned} \langle \tilde{K}(\tau) \rangle &= \tau^2 \langle \tilde{D}(\tau) \rangle \\ &= \tau^2 \left[\langle D^0(\tau) \rangle + \langle D^1(\tau) \rangle \right] , \end{aligned} \quad (48)$$

and that $\langle D^1(\tau) \rangle$ is asymptotic and $\langle D^0(\tau) \rangle$ is not asymptotic in the evolution of $\bar{P}(\tau, k)$ from (45),

Now we can calculate α from (42b), by substituting for \bar{P} from (47). We find that

$$\alpha(\omega, k) = \frac{c}{\alpha} \omega'^3 M(\omega, k) , \quad (49a)$$

with

$$\alpha = \alpha_1 + i\alpha_2 , \quad (49b)$$

is controlled by the memory-loss function

$$M(\omega, k) = \int_0^\infty d\tau \tau^2 e^{i\Omega\tau} e^{-\omega_D^3 \tau^3 - m^0 \tau^4} , \quad (50a)$$

which determines the life-time of memory in the form

$$\tau_M = M^{1/3} . \quad (50b)$$

The characteristic frequencies are

$$\Omega = \omega - \underline{k} \cdot \underline{v} , \quad \omega'_D = (c \langle D^1 \rangle k^2)^{1/3} , \quad m^0 = (c^0 \langle E^{02} \rangle k^2)^{1/4} , \quad (51a)$$

and the numerical coefficients are

$$\frac{c}{\alpha} = \pi/c^1 , \quad c' = (1/3)^{1/3} , \quad c^0 = (1/8)^{1/4} \quad (51b)$$

As seen from (43) and (44), the collective collision C^0 is related to the pair collision C^0 by a factor β , which can be rewritten as

$$\beta(\omega, k) = [(1 - \alpha_1)^2 + \alpha_2^2]^{-1}, \quad (51)$$

in terms of the real and imaginary parts of α , by (49). Hence the determination of (51) rests on the calculation of the memory integral (50a) and its decomposition into its real and imaginary parts. The calculation is made by an interpolation over the three regions dominated by Ω , ω_D' and m^0 , separately. The results are given as follows:

(a) For weak turbulence, i.e. $\Omega \gg \omega_D', m^0$, we find a shielding in the collision by a factor

$$\beta \approx 1 - 2 c_s (\omega_D' / \Omega)^3 \approx 1. \quad (52)$$

(b) For strong turbulence, i.e. $\Omega \ll \omega_D', m^0$, we find an enhancement in the collision by a factor

$$\beta \approx 12.51, \quad \text{if } m^0 \ll \omega_D' \quad (53a)$$

$$[1 + 2.77 (\omega_D' / m^0)^3] \approx 1, \text{ if } m^0 \gg \omega_D'. \quad (53b)$$

We can approximate β by its constant asymptotic values which are :

(a) weak turbulence, $\Omega \gg \omega_D', m^0$

$$\beta \approx 1, \quad (54a)$$

(b) strong turbulence, $\Omega \ll \omega_D', m^0$

$$\beta \approx \begin{cases} 1, & \text{for } \omega_D' \ll m^0 \text{ (large } k) \\ 2.51, & \text{for } \omega_D' \gg m^0 \text{ (small } k) \end{cases} \quad (54b)$$

$$2.51, \text{ for } \omega_D' \gg m^0 \text{ (small } k) \quad (54c)$$

It entails the simplification

$$c^0(t, \underline{x}, \underline{v}) \approx \beta \mathcal{C}^0(t, \underline{x}, \underline{v}) \quad (55)$$

of (43),

VII. SPECTRAL BALANCE

The kinetic equation (18) of f^0 , rewritten as

$$(\partial_t + A^0 L) f^0 = -L^0 \bar{f} + c^0 \quad (56)$$

has a collective collision which can be written as

$$c^0 = \beta \langle \mathcal{D}' \rangle \cdot \partial \{ f^0(t-\tau) \}, \quad (57)$$

with a diffusivity found by (55b),

Consider now a homogeneous turbulence, i.e., with $\bar{\underline{u}} = 0$, $\bar{\underline{E}} = 0$. By taking the first moment of the kinetic equation (56), we obtain the following equation of macro-momentum:

$$(\partial_t + A^0 \underline{u} \cdot \nabla - \nu \nabla^2) \underline{u}^0 = \underline{E}^0 + \underline{Q}^0, \quad (58)$$

with a collision

$$\underline{Q}^0 = \beta \int d\underline{v} \underline{v} \cdot \langle \mathcal{D}' \rangle \cdot \partial \{ f^0(t-\tau) \}, \quad (59)$$

by (57). Subsequently, we take the curl of (58) and (59), to obtain the following equation of macro-vorticity of the two-dimensional geostrophic turbulence:

$$(\partial_t + \underline{u} \cdot \nabla - \nu \nabla^2) \zeta^0 = \nabla \times \underline{E}^0 + \nabla \times \underline{Q}^0 \quad (60)$$

Finally, upon multiplying (60) by ζ^0 and averaging, we find the equation of spectral balance in the form:

$$\frac{1}{2} \partial_t \langle \xi^{\circ 2} \rangle = -T_{\xi}^{\circ} - \epsilon_{\xi}^{\circ} \quad (61)$$

with the following transport functions:

$$\text{cascade transfer} \quad T_{\xi}^{\circ} = \langle \xi^{\circ} \cdot \nabla \times Q^{\circ} \rangle \quad (62a)$$

$$\text{molecular dissipation} \quad \epsilon_{\xi}^{\circ} = \nu \langle (\nabla \xi^{\circ})^2 \rangle \quad (62b)$$

Note that the field

$$\underline{E}^{\circ} = \underline{E}_P^{\circ} + \underline{E}_X^{\circ}$$

consists of two parts: The first part has a zero vorticity by (8), and the second part, which is a random source, does not correlate with ξ° .

Thus the field \underline{E}° does not appear in the spectral balance in the form of the correlation $\langle \xi^{\circ} \cdot \underline{E}^{\circ} \rangle$. But it governs the collision

$$Q^{\circ} = Q_P^{\circ} + Q_X^{\circ}, \quad (63a)$$

with

$$Q_P^{\circ} = \beta \int d\underline{v} \underline{v} \cdot \langle \underline{D}_P' \rangle \delta \{ f^{\circ}(t-\tau) \}, \quad Q_X^{\circ} = \beta \int d\underline{v} \underline{v} \cdot \langle \underline{D}_X' \rangle \delta \{ f^{\circ}(t-\tau) \}, \quad (63b)$$

through the diffusivities

$$\langle \underline{D}' \rangle = \langle \underline{D}_P' \rangle + \langle \underline{D}_X' \rangle \quad (63c)$$

$$\text{with} \quad \langle \underline{D}_P' \rangle = \int_0^{\infty} d\tau \langle \underline{E}_P'(t, \underline{x}) \hat{U}(t, t-\tau) \underline{E}_P'(t-\tau) \rangle \cdot \langle \underline{D}_X' \rangle = \int_0^{t_0} d\tau \langle \underline{E}_X'(t, \underline{x}) \hat{U}(t, t-\tau) \underline{E}_X'(t-\tau) \rangle. \quad (63d)$$

By separating the collision into two parts as in (63a), we find the transfer function also separated into two parts, as follows:

$$T_{\xi}^{\circ} = T_P^{\circ} + T_X^{\circ}, \quad (64a)$$

with

$$T_{Po} = \langle \xi^{\circ} \cdot \nabla \times Q_P^{\circ} \rangle, \quad T_{Xo} = \langle \xi^{\circ} \cdot \nabla \times Q_X^{\circ} \rangle. \quad (64b)$$

In the following, we shall follow the same method of treatment of

the transfer function as developed earlier by Tchen [1981]. As notations we introduce the spectral functions $F_u(k)$ and $F_\zeta(k)$, and the spectral density $\langle s_X'(k) \rangle$, such that

$$\langle u^{o2} \rangle = 2 \int_0^k dk' F_u(k'), \quad \langle \zeta^{o2} \rangle = 2 \int_0^k dk' F_\zeta(k'), \quad \langle E_X^{o2} \rangle = d \int dk' \langle s_X^{o2}(k') \rangle, \quad (65a)$$

and the second spectral moments

$$R_\zeta^o = 2 \int_0^k dk' k'^2 F_\zeta(k'), \quad R_u^o = 2 \int_0^k dk' k'^2 F_u(k'), \quad (65b)$$

with the relationship

$$F_\zeta(k) = d^{-1} k^2 F_u(k), \quad (65c)$$

by definition (3) in $d=2$ dimensions. By omitting the details of calculation, we find the results as follows:

(a) Cascade-transfer by the E_p - fluctuations

The transfer function is obtained in the form:

$$T_p^o = \begin{cases} \beta K_u' R_\zeta^o, & \text{for large } k \end{cases} \quad (66a)$$

$$\begin{cases} \beta \Omega_u' \langle \zeta^{o2} \rangle, & \text{for small } k. \end{cases} \quad (66b)$$

The coefficient β represents the effect of the multiple collision (54). The transfer function is seen to be governed by the eddy viscosity

$$\langle K_u' \rangle = \frac{2}{d} \int_k^\infty dk' F_u(k'') \left\{ \left[\frac{4}{\pi d^2} k'^3 F_u(k'') \right]^{1/2} + k''^2 \Omega_u' \right\}^{-1}$$

which has an asymptotic value

$$\langle K_u' \rangle = \sqrt{\pi} \int_k^\infty dk'' \left(F_u(k'') / k''^3 \right)^{1/2} \quad (67)$$

for large k , and by the rate of damping

$$\Omega_u' = \langle K_u' \rangle \{k''^2\}, \quad (68)$$

with

$$\langle K_u' \rangle = \langle K_u' \rangle = \left[\frac{4}{d} \int_k^\infty dk'' k''^{-2} F_u(k'') \right]^{1/2} \quad (69a)$$

for small k . Note that (69a) is the solution of

$$\langle K_u' \rangle = \frac{2}{d} \int_k^\infty dk'' \frac{F_u(k'')}{k'^2 \langle K_u' \rangle} \quad (69b)$$

The asymptotic formulas (67) and (69a) agree with our earlier cascade theory [Tchen, 1973, 1978] and other dimensional theories [Heisenberg, 1948; Gisinu, 1969].

(b) Cascade-transfer by E_X - fluctuations

The transfer function is obtained in the form

$$T_X^0 \approx \beta \langle K_X' \rangle R_\zeta^0, \quad \text{for large } k \quad (70a)$$

$$\beta \Omega_X' \langle \zeta'^2 \rangle, \quad \text{for small } k, \quad (70b)$$

and is governed by the eddy viscosity

$$\langle K_X' \rangle = \int dk'' \langle s_X'(k'') \rangle G(k'') \quad (71)$$

for large k , and the rate coefficient of damping

$$\Omega_X' = \int dk'' k''^2 \langle s_X'(k'') \rangle G(k'') \quad (72)$$

for small k . Here the modulation function

$$G(k'') = \left[\frac{1}{3} \Gamma\left(\frac{7}{4}\right) m^0(k'') + 2^{-1/3} \omega_k'(k'') \right]^{-3}, \Gamma\left(\frac{7}{4}\right) = 0.919 \quad (73)$$

is characterized by the two frequencies as follows:

$$m^0(k'') = \left[\frac{1}{8d} k''^2 \int dk_1'' \langle s_X^0(k_1'') \rangle \right]^{1/4}, \quad \omega_k'(k'') = k''^2 \langle K_X'(k'') \rangle \quad (74)$$

It is to be noted that the simplified formulas of all the transfer functions listed above are characteristically in the form of products of two functions. A general formula not in the form of a product has also been developed by Tchen (1981b), but is too complicated for the application to the present problem.

We first consider the portion of the spectrum of large k . The small eddies are embedded in a gradient of big eddies and in their coupling with the bigger eddies, the small ones play the role of an eddy viscosity, to cause a loss of the bigger ones at the rate $\partial_t \langle \epsilon^2 \rangle < 0$. This transfer is in the pattern of a cascade from the big eddies to the smaller ones. It is of the gradient type and is called the direct cascade. Now we consider the portion of the spectrum of small k . There the big eddies have no more gradient left in the medium in which they evolve, if the turbulent medium is isotropic. The coupling between the big and small eddies is not through the gradient transfer, but is of the type of the damping. The rate coefficients $\Omega'_\mu, \Omega'_\chi$ are originated from the wave-particle interaction [Tchen, 1981], and may become negative when k is sufficiently small. Then they give an amplification. When this happens, the transfer functions reverse their roles from a loss into a production, i.e. $\partial_t \langle \epsilon^2 \rangle > 0$. Such a transfer mechanism is called the reverse cascade. An analogous phenomenon is known in plasmas as the Landau damping or amplification.

If the demarcation wavenumber between the direct and reverse cascades is k_{rev} , the large and small values of k refer to

$$k > k_{\text{rev}} \quad \text{and} \quad k < k_{\text{rev}}, \quad (75)$$

respectively.

VIII. STATISTICAL EQUILIBRIUM AND NON-EQUILIBRIUM

By differentiating the equation of spectral balance (61) with respect to k , we obtain:

$$\partial_t F_{\zeta}(k) = -\dot{\xi}_{\zeta}^0 - \dot{T}_{\zeta}^0, \quad (\dot{} = \partial/\partial k). \quad (76)$$

The left hand side vanishes in a statistical equilibrium, yielding

$$\xi_{\zeta}^0 + T_{\zeta}^0 = \xi_{\zeta}. \quad (77)$$

The constant of integration in the right hand side is determined by the condition at $k = \infty$, i.e.,

$$\xi_{\zeta}^0 = \xi_{\zeta} \equiv \nu \langle (\nabla \zeta)^2 \rangle \text{ and } T_{\zeta}^0 = 0. \quad (78)$$

In addition, we note that

$$T_{\zeta}^0 = 0, \quad \text{at } k=0. \quad (79)$$

For the inertial subrange in statistical equilibrium, we can write the spectral balance in the form

$$T_{\zeta}^0 = T_P^0 + T_X^0 = \xi_{\zeta} \quad (80)$$

by omitting the viscous dissipation ξ_5^0 by definition of the inertia subrange. Here ξ_{ζ} is a sink in the enstrophy transfer across the spectrum. The statistical equilibrium requires that the two transfer functions find a net positive transfer to balance ξ_{ζ} .

On the other hand, if the net transfer is negative, ξ_{ζ} ceases to be a useful parameter, and the hypothesis of the statistical equilibrium becomes invalid. Consequently, we have to return to the original equation (61) for the spectral balance in non-equilibrium:

$$\frac{1}{2} \partial_t \langle \zeta^2 \rangle = -T_{\zeta}^0 \quad (81)$$

by assuming a supply to the spectrum from an non-stationary source of larger scale,

Here we have again omitted the molecular dissipation. The negative net transfer, i.e. $T_{\zeta}^0 < 0$, means an amplification of the enstrophy in time, and forms a reverse cascade toward the small wavenumbers in the spectrum.

IX. SPECTRAL DISTRIBUTIONS

In the preceding Section, we have distinguished between a direct cascade and a reverse cascade, as characterized by a transfer of the gradient type at large k and by a transfer of the damping type at small k . We

analyse the spectral distributions for these two cascades in the following lines.

A. Direct Cascade

We consider the joint enstrophy transfer by u and E_X = fluctuations in the spectral balance (80), and rewrite it as

$$[\langle K_u \rangle + \langle K_X \rangle] \int_0^k dk' k'^2 F_{\zeta}(k') = \epsilon_{\zeta}, \quad (82)$$

with the use of (66a), (67), (70a) and (71). The eddy viscosity $\langle K_X \rangle$ governs the transfer under the driving force exclusively, and the eddy viscosity $\langle K_u \rangle$ governs the transfer without the driving force.

An approximate solution of (82) can be obtained by an interpolation of the following two separate equations of spectral balance:

$$2 \langle K_u \rangle \int_0^k dk' k'^2 F_{\zeta}(k') = \epsilon_{\zeta} \quad (83a)$$

$$2 \langle K_X \rangle \int_0^k dk' k'^2 F_{\zeta}(k') = \epsilon_{\zeta}, \quad (83b)$$

finding the asymptotic solution

$$F_u(k) = C \epsilon_{\zeta}^{2/3} k^{-3}, \quad C \approx 2.6 \quad (84a)$$

of (83a), with

$$\langle K_u' \rangle = b_K \xi^{1/3} k^{-2}, \quad b_K = 2/C = 0.77, \quad (84b)$$

and the asymptotic solution

$$F_u(k) = A_u \xi \langle s \rangle^{-1/4} k^{-4}, \quad A_u = c_K^{-1} \approx 0.60 \quad (85a)$$

of (83b) with

$$\langle K_X' \rangle = c_K \langle s \rangle^{1/4} k^{-1}, \quad c_K \approx 1.66. \quad (85b)$$

Note that, for obtaining the solution (85) of the equation of balance (83b), the eddy viscosity (71) has been written as

$$\begin{aligned} K_X' &= \int d\mathbf{k}'' \langle s_X'(\mathbf{k}'') \rangle G(\mathbf{k}') \\ &= \langle s \rangle \int_k^\infty d\mathbf{k}'' 2\pi k'' G(\mathbf{k}''), \quad \text{for finite } k, \end{aligned} \quad (86)$$

where $\langle s \rangle$ is the trace of the spectral density tensor, i.e.,

$$\langle s \rangle = \text{trace} \langle s_X' \rangle. \quad (87)$$

As a white noise it has the form

$$\begin{aligned} \langle s_X(\mathbf{k}'') \rangle &= (s), \quad \text{independent of } k'', \quad \text{for finite } k'' \\ &= 0, \quad \text{for } k'' = \infty. \end{aligned} \quad (88)$$

Also note that

$$d\mathbf{k}'' = 2\pi k'' d\mathbf{k}'' \quad (89)$$

We have calculated $G(\mathbf{k}'')$ from (73) and (74) for substitution into (86), to find $\langle K_X' \rangle$ as was written in (85b). The numerical coefficient c_K has been determined from the equation:

$$c_K = 2\pi \left[\frac{1}{3} \Gamma\left(\frac{7}{4}\right) \left(\frac{\pi}{8}\right)^{1/4} + 2^{-1/3} c_K \right]^{-3} \quad (90)$$

Finally by substituting for $\langle k_X' \rangle$ thus obtained, we have found (85a) as the solution of (83b).

Now we return to the problem of the interpolation of the two asymptotic solutions (84a) and (85a), by incorporating the two eddy viscosities (84b) and (85b) in the equation (82) of the joint balance, which becomes :

$$(b_K \xi_\zeta^{1/3} k^{-2} + c_K \langle s \rangle^{1/4} k^{-1}) 2 \int_0^k dk' k'^2 F(k') = \xi_\zeta. \quad (91)$$

The equation (91) shows that the dominant eddy viscosities are in the form of

$$\langle k_u' \rangle \text{ for } k < k_0, \text{ and } \langle k_X' \rangle \text{ for } k > k_0 \quad (92)$$

Then we find the following formula of interpolation:

$$F_u(k) = C \xi_\zeta^{2/3} k^{-3} \psi(k/k_0), \quad (93)$$

with

$$\psi(k/k_0) = \frac{1}{2} \left[\frac{1}{(1+k/k_0)^2} + \frac{1}{1+k/k_0} \right] \quad (94)$$

and

$$\begin{aligned} k_0 &= 2(A_u/C) \xi_\zeta^{1/3} \langle s \rangle^{-1/4} \\ &\approx 0.46 \langle s \rangle^{-1/4} \end{aligned} \quad (95)$$

It is easy to verify that the general solution (93) includes the asymptotic solutions (84a) and (85a) for $k < k_0$ and $k > k_0$, respectively. It is seen that the spectral law k^{-3} precedes the spectral law k^{-4} in the sequence of increasing wavenumbers.

B. Reverse Cascade

For the reverse cascade which appears in the region of small wavenumbers,

we shall use the formulas (66b), (68) and (69), and write the spectral balance as

$$-\beta(\dot{\mathcal{N}}_u' + \dot{\mathcal{N}}_x') \langle \zeta^2 \rangle = \varepsilon_* \quad (96a)$$

or

$$-d^{-1} \beta(\dot{\mathcal{N}}_u' + \dot{\mathcal{N}}_x') R_u^0 = \varepsilon_* \quad (96b)$$

by a change of notations from (65b) and (65c). Here ε_* is the rate of increase of

$$\frac{1}{2} \partial_t \langle \zeta^2 \rangle \cong \varepsilon_* \quad (97)$$

in non-equilibrium, and is assumed to be independent of k in the global balance which may include the supply from the synoptic scales.

We divide both sides of (96b) by R_u^0 and differentiate, to obtain

$$\beta d^{-1} (\dot{\mathcal{N}}_x' + \dot{\mathcal{N}}_u') = \varepsilon_* \sigma / k R_u^0 \quad (98)$$

or

$$R_u^0 = \sigma \beta^{-1} d \varepsilon_* (\dot{\mathcal{N}}_x' + \dot{\mathcal{N}}_u')^{-1} k^{-1}, \quad (99)$$

where

$$\sigma \equiv (k \dot{R}_u^0 / R_u^0) \simeq 1 \quad (100)$$

is seen to be a positive and dimensionless number in this subrange, since R_u^0 is a positive function which increases with k monotonically, by definition (68b). We can estimate σ to be of the order of unity.

Finally we differentiate (99) with respect to k and find the spectrum

$$F_u(k) = A_u^* \varepsilon_* \langle s \rangle^{-1/4} \psi_*(k) k^{-4}, \quad A_u^* = (d\sigma/2\beta\sigma_*) c_K^{-1} = 0.24, \quad (101)$$

where

$$\psi_* = \frac{k/k_0}{1 + k/k_0}, \quad k_0 = (A_u^*/C^*) \epsilon_*^{1/3} \simeq 0.22 \epsilon_*^{1/3}, \quad (102a)$$

and

$$\sigma_* \equiv -k \dot{\Omega}_u' / \Omega_u'$$

is estimated to be

$$\sigma_* \simeq 2, \quad (102b)$$

if we approximate $\Omega_u' \simeq k^2 \langle K_u \rangle$. Use of (68) and (69) has been made. Since the reverse cascade occurs at small k , we have taken the value (54c) for β .

The solution (101) takes the following asymptotic expressions

$$F_u(k) = \begin{cases} C^* \epsilon_*^{2/3} k^{-3}, & \text{for } k < k_0 \end{cases} \quad (103a)$$

$$\begin{cases} A_u^* \epsilon_* \langle s \rangle^{-1/4} k^{-4}, & \text{for } k > k_0, \end{cases} \quad (103b)$$

with

$$C^* = \beta^{-2/3} \alpha (2\sigma/\sigma_*)^{2/3} \simeq 1.08, \quad A_u^* \simeq 0.24. \quad (104)$$

We note that the numerical coefficients (104) in the reverse cascade are smaller than their corresponding values (84a) and (85a) in the direct cascade.

X. CONCLUSION

In the direct cascade, a critical wavenumber k_0 separates the spectrum into two regions: The region at $k > k_0$ shows a dominant role of the random force, and has a power law k^{-4} . The region at $k < k_0$ has a negligible effect of the random force, and yields a power law k^{-3} . In the reverse cascade, the same power laws repeat at a new critical wavenumber k_0^* ($< k_0$), with their numerical coefficients modified by the multiple collision, and with different parameters. A bump appears in the transition from the direct cascade into the inverse cascade [Weinstock, 1978].

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SECTION 3

EQUIVALENT METHODS

FOR QUASILINEAR TURBULENT TRAJECTORIES

J. H. Misguich and C. M. Tchen*

ABSTRACT.

The propagator formalism is summarized in a self-consistent way for the asymptotic quasi-linear equation.

A comparison is performed between the propagators and the Green's functions in the case of the non-asymptotic quasi-linear equation. This allows to prove the equivalence of both kind of approximations used to describe perturbed trajectories of plasma turbulence.

*This work investigates the detailed dynamics of the perturbed trajectory in turbulence. The manuscript is prepared for publication.

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Renormalization introduced in the microscopic theories of plasma and fluid turbulence essentially consists in taking into account the turbulent perturbation in particle trajectories.

These turbulent modification (perturbed orbits) can of course be calculated by integrating the equations of motion in the fluctuating field. In a kinetic theory one uses however more compact and powerful tools which allow to directly describe the time-evolution of observable averages of all dynamical function by means of a unique entity: the propagator acting on phase variables \underline{x} and \underline{v} , or the Green function.

The aim of this work consists to prove the equivalence between two approximations which have been developed independently in these two formalisms. This allows us to bridge the gap between different methods used by various authors.

Turbulent modifications of the trajectories are described in the lowest approximation as quadratic function of the fluctuating field, and by considering all other trajectories as unperturbed by the turbulent field. This approximation has also been used in the quasi-linear equation for the distribution function and is thus referred to as the quasi-linear approximation for the trajectories (although it uses free trajectories as basic ones) .

The turbulent trajectories so-obtained are then used as a basic ingredient in the renormalized theory of plasma or fluid turbulence in the next approximation which is referred to as the Renormalized Quasi-Linear approximation (RQL) ^{1,2,3,4,5,6} / This latter appears to be equivalent to the Direct Interaction Approximation (DIA) introduced in fluid turbulence ^{4,3} / . Thus the renormalized propagator appearing in this approximation (in the weak-coupling limit) actually describes lowest order turbulent trajectories (i.e. in the quasi-linear approximation) and

we will limit ourselves to this approximation for the particle trajectories.

About the methods used to describe these trajectories, we have to remark that one has first integrated the equations of motion /1/. WEINSTOCK /2/ has then introduced in this problem mathematical operators acting on the phase variables \underline{x} and \underline{v} , the propagators // the role of which consists to describe the time-evolution of associated dynamical variables $\underline{x}(t)$ and $\underline{v}(t)$, and, consequently, (Liouville's theorem) to describe the time-evolution of distribution functions. In order to calculate the action of these propagators, WEINSTOCK had however to introduce particle trajectories which have been calculated by integrating the equations of motion. Some subtle effects, like that giving rise to an average displacement of the particles have been first missed by this method /3/.

Other unexpected effects, like the non-vanishing correlation between particle velocity and position, have been found due to the first explicit calculation of the turbulent propagator /8/. In Ref. 9 and 10 we have discussed the explicit relation between trajectories and the propagator for forward and backward propagation in time.

Other authors have based the kinetic theory on the Green functions /6/. We will compare here these methods and prove the equivalence between the approximations used. In spite of the global equivalence a few differences remain, namely concerning non-Markovian or memory effects.

II. KINETIC QUASI-LINEAR EQUATION

II.A. Method of derivation

The renormalized propagator and Green's function which will be derived are described by the quasi-linear equation which will be deduced here in a simple way.

For electrostatic plasma turbulence, as well as for Navier-Stokes fluid turbulence, the starting point can be put in the form of the plasma Klimontovich equation

$$\left[\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{E}(\underline{x}, t) \cdot \frac{\partial}{\partial \underline{v}} \right] N(\underline{x}, \underline{v}, t) = 0 \quad \text{II.1}$$

(in the plasma case a ratio q/m is included in the electric field \underline{E} , with q the charge and m the mass of the particle). In the plasma case $N(\underline{x}, \underline{v}, t)$ is the Klimontovich microscopic distribution

$$N(\underline{x}, \underline{v}, t) = \sum_{i=1}^N \delta[\underline{x} - \underline{x}_i(t)] \delta[\underline{v} - \underline{v}_i(t)] \quad \text{4.2}$$

Here \underline{x} and \underline{v} are the phase space variables, while $\underline{x}_i(t)$ and $\underline{v}_i(t)$ are the dynamical (time-dependent) variables described by the exact equations of motion. The electric field is the microscopic one given in term of N by the Poisson equation /11,12/.

The ensemble average introduced in statistical mechanics allows one to define the usual distribution function by

$$\bar{f}(\underline{x}, \underline{v}, t) = \langle N(\underline{x}, \underline{v}, t) \rangle \quad \text{II.3}$$

and the Vlasov mean field for inhomogeneous systems by $\langle \underline{E}(\underline{x}, t) \rangle$

Fluctuations are defined as the difference with the average :

$$f' = N - \langle N \rangle \quad \text{II.4}$$

$$E' = E - \langle E \rangle \quad \text{II.5}$$

The Klimontovich equation can be written in the compact form $(\partial_t \equiv \frac{\partial}{\partial t})$

$$\partial_t N = L N \quad \text{II.6}$$

where $L = \bar{L} + L'$

with

$$\begin{aligned} \bar{L} &\equiv \langle L \rangle = -\gamma \cdot \nabla - \langle \underline{E}(x,t) \rangle \cdot \frac{\partial}{\partial \underline{v}} \\ L' &= -\underline{E}'(x,t) \cdot \frac{\partial}{\partial \underline{v}} \end{aligned} \quad \text{II.7.}$$

Separation in average and fluctuating equations gives

$$\begin{cases} \partial_t \bar{F} = \bar{L} \bar{F} + \langle L' F' \rangle & \text{II.8} \\ \partial_t F' = \bar{L} F' + L' \bar{F} + \mathcal{B} L' F' & \text{II.9a} \end{cases}$$

$$\begin{aligned} \text{where } \mathcal{B} L' F' &= L' F' - \langle L' F' \rangle \equiv (1-A) L' F' & \text{II.10} \\ A \dots &= \langle \dots \rangle \end{aligned}$$

describes the fluctuating part of the product. Let us remark that this last equation (II.9) also writes

$$\partial_t F' = L F' + L' \bar{F} - \langle L' F' \rangle \quad \text{II.9 b.}$$

From the two equations (II.8) and (II.9) a general kinetic equation can be derived in the following way : equation (II.9) is solved for F' in terms of \bar{F} and the solution is substituted into (II.8). A closed equation is then obtained for \bar{F} , which still contains the initial fluctuation $F'(t_0)$: this equation is called the general master equation. This equation, obtained by Weinstock from the Vlasov equation, is fully analogous to the master equation obtained by Prigogine-Résibois in statistical mechanics, or to the one obtained by Mori or Zwanzig.

In order to go to a kinetic equation, irreversibility has to be introduced in some way : it usually consists to take the limit of long times $(t-t_0)$ compared to microscopic elementary

times (Eulerian correlation time of fluctuating fields) and to neglect the term involving the initial fluctuation.

One of us has developed with BALESCU a theory which shows that this general kinetic equation can be obtained in a rigorous way for one ("kinetic") projection of the distribution function, due to the existence of two independent subdynamics /13,14/. This theory brings justification and necessary validity conditions for the simple derivation which is made here.

II.B. Propagators.

The basic tool for solving equations (I.9a,b) is the propagator. We will first present the simple and trivial example of the free propagator associated with unperturbed particle trajectories. In such a simple case, the evolution equation reduces to

$$\partial_t F = L_0 F$$

where

$$L_0 = -\underline{v} \cdot \underline{\nabla}$$

The solution immediately writes

$$F(\underline{x}, \underline{v}, t) = e^{L_0 t} F(\underline{x}, \underline{v}, 0)$$

Here the exponential operator

$$e^{L_0 t} = U_0(t, 0) = e^{-t \underline{v} \cdot \underline{\nabla}}$$

which is solution of the same equation

$$\partial_t U_0(t, 0) = L_0 U_0(t, 0)$$

allows us to describe the time-evolution of the distribution function as

$$F(\underline{x}, \underline{v}, t) = U_0(t, 0) F(\underline{x}, \underline{v}, 0)$$

It is important to remark that this same operator also describes particle trajectories:

$$\underline{x}(t+\tau) = U_0(t, t+\tau) \underline{x} = e^{\tau \underline{v} \cdot \nabla} \underline{x} = \underline{x} + \tau \underline{v}$$

$$\underline{v}(t+\tau) = U_0(t, t+\tau) \underline{v} = e^{\tau \underline{v} \cdot \nabla} \underline{v} = \underline{v}$$

and this allows one to check the "Liouville theorem" in its simple form

$$\begin{aligned} F(\underline{x}, \underline{v}, t) &= e^{-t \underline{v} \cdot \nabla} F(\underline{x}, \underline{v}, 0) = F[\underline{x} - \underline{v}t, \underline{v}, 0] \\ &= F[\underline{x}(-t), \underline{v}(-t), 0] \end{aligned}$$

This is the basic advantage of the propagators: the same mathematical tool allows one to describe the forward propagation of the distribution function in time:

$$F(\underline{x}, \underline{v}, t+\tau) = U_0(t+\tau, t) F(\underline{x}, \underline{v}, t)$$

and the inverse (backward) evolution of the dynamical variables:

$$\underline{x}(t) = U_0(t+\tau, t) \underline{x}(t+\tau)$$

thus

$$\underline{x}(t+\tau) = [U_0(t+\tau, t)]^{-1} \underline{x} = U_0(t, t+\tau) \underline{x}$$

Detailed demonstrations have been given in the general case /16/

In the case of equation II.9 one can define on one hand the propagator $U(t, t_0)$ describing the exact motion of the particles in the fluctuating field \underline{E}' or \underline{L}' :

$$\partial_t U(t, t_0) = \underline{L}(\underline{x}, t) U(t, t_0) \quad \text{II.10}$$

(U operates on both \underline{x} and \underline{v})

or, on the other hand, the Weinstock propagator associated with the homogeneous part of equation (II.9a) :

$$\partial_t \Lambda(t, t_0) = [\underline{L}(t) + \mathcal{B} \underline{L}'(t)] \Lambda(t, t_0) \quad \text{II.11}$$

Here we have to stress the fact that the fluctuation operator \mathcal{B} equally acts on L' and on Λ . The solution of equations (II.9a,b) can thus be written

$$F'(t) = A(t, t_0) F'(t_0) + \int_{t_0}^t dt' \Lambda(t, t') L'(t') \bar{F}(t') \quad \text{II.12a}$$

$$= U(t, t_0) F'(t_0) + \int_{t_0}^t dt' U(t, t') L'(t') \bar{F}(t') - \int_{t_0}^t dt' U(t, t') \langle L'(t') F'(t') \rangle \quad \text{II.12b.}$$

where $L'(t) = -E(x, t') \cdot \frac{\partial}{\partial v}$ at point x

II.C. General kinetic equation

The kinetic limit of long times can be justified by the subdynamics method /13,14/; it consists in neglecting the influence of the initial fluctuation $F'(t_0)$ and in taking the asymptotic long time limit $t_0 \rightarrow -\infty$. In this case we have

$$F'(t) \rightarrow \int_0^\infty dz \Lambda(t, t-z) L'(t-z) \bar{F}(t-z) \quad \text{II.13}$$

which describes the general mechanism of creation of fluctuations from the average function, by means of the fluctuating fields.

On equation (II.12b) this limit gives

$$C(t) \equiv \langle L'(t) F'(t) \rangle = \int_0^\infty dz \langle L'(t) U(t, t-z) L'(t-z) \rangle \bar{F}(t-z) - \int_0^\infty dz \langle L'(t) U(t, t-z) \rangle C(t-z) \quad \text{II.14}$$

which remains an integral equation. By decomposing

$$U = \bar{U} + U' \quad \text{II.15}$$

we have

$$C(t) = \int_0^\infty dz \langle L'(t) \bar{U}(t, t-z) L'(t-z) \rangle \bar{F}(t-z) + \int_0^\infty dz \langle L'(t) U'(t, t-z) L'(t-z) \rangle \bar{F}(t-z) - \int_0^\infty dz \langle L'(t) U'(t, t-z) \rangle C(t-z) \quad \text{II.16.}$$

Since we have

$$\partial_t \bar{U}(t, t') = \langle L(t) U(t, t') \rangle = \bar{L}(t) \bar{U}(t, t') + \langle L'(t) U'(t, t') \rangle \quad \text{II.17}$$

the last term also writes as follows

$$C(t) = \int_{-\infty}^t dt' \langle L'(t) U(t,t') L'(t') \rangle \bar{F}(t') - \int_{-\infty}^t dt' [\frac{\partial}{\partial t} - \bar{L}(t)] \bar{U}(t,t') C(t') \quad \text{II.18}$$

By using the complete (but formal) solution (II.12a) in I.8 we obtain the Weinstock master equation

$$\partial_t \bar{F}(t) = \bar{L}(t) \bar{F}(t) + \int_{t_0}^t dt' \langle L'(t) \Lambda(t,t') L'(t') \rangle \bar{F}(t') + \langle L'(t) \Lambda(t,t_0) F'(t_0) \rangle \quad \text{II.19}$$

the kinetic limit of which gives the general kinetic equation

$$\partial_t \bar{F}(t) = \bar{L}(t) \bar{F}(t) + \int_0^\infty d\tau \langle L'(t) \Lambda(t,t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \quad \text{II.20}$$

in an explicitly non-Markovian form ($\partial_t \bar{F}(t)$ depends on $\bar{F}(t-\tau)$ at earlier times). The second term in the r.h.s. represents the turbulent collision operator in its most general form.

The subdynamics method allows one to directly obtain the same equation in an apparently Markovian form (including however all the non-Markovian effects : see /16/):

$$\begin{aligned} \partial_t \bar{F}(t) &= \bar{L}(t) \bar{F}(t) + \int_0^\infty d\tau \langle L'(t) \Lambda(t,t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \bar{F}(t) \\ &= [\bar{L}(t) + G(t)] \bar{F}(t) \end{aligned} \quad \text{II.21}$$

These two forms are rigorously equivalent since the turbulent propagator \bar{V} is defined as the propagator of the kinetic equation itself:

$$\partial_t \bar{V}(t, t_0) = [\bar{L}(t) + G(t)] \bar{V}(t, t_0) \quad \text{II.22}$$

thus

$$\bar{F}(x, y, t) = \bar{V}(t, t_0) \bar{F}(x, y, t_0) \quad \text{II.23}$$

Equation II.22 remains a non-linear equation for the turbulent propagator. This dynamical non-linearity can be expressed also in the non-linear equation for the turbulent collision operator

$$G(t) = \int_0^\infty dz \langle L'(t) \Lambda(t, t-z) L'(t-z) \rangle \bar{V}(t-z, t) \quad \text{II.24}$$

where \bar{V} is a superoperator, non-linear in the operator G :

$$\bar{V}(t, t_0) = X_+ \exp \int_{t_0}^t dt' [\bar{L}(t') + G(t')] \quad \text{II.25}$$

This kind of operators have been studied in detail in Ref./15/.

The formal solution of (II.22) can be written in terms of the free propagator in the average field

$$\partial_t U^0(t, t_0) = \bar{L}(t) U^0(t, t_0) \quad \text{II.26}$$

in the form of the DYSON equation /18/

$$\begin{aligned} \bar{V}(t, t_0) &= U^0(t, t_0) + \int_{t_0}^t dt' U^0(t, t') G(t') \bar{V}(t', t_0) \\ &= U^0(t, t_0) + \int_{t_0}^t dt' U^0(t, t') \int_0^\infty dz \langle L'(t') \Lambda(t', t'-z) L'(t'-z) \rangle \bar{V}(t'-z, t_0) \end{aligned} \quad \text{II.27}$$

where G is given by II.24 in terms of Λ and \bar{V} . The renormalized quasi-linear approximation (RQL) which is equivalent to the Direct Interaction Approximation (DIA) introduced by Kraichnan, consists in approximating the Weinstock propagator Λ by the average turbulent propagator \bar{V} . One obtains in this way:

$$\bar{V}_{\text{RQL}}(t, t_0) = U^0(t, t_0) + \int_{t_0}^t dt' U^0(t, t') G_{\text{RQL}}(t') \bar{V}_{\text{RQL}}(t', t_0) \quad \text{II.28}$$

where the RQL turbulent collision operator is defined by the following non-linear equation in terms of \bar{V}_{RQL} :

$$G_{\text{RQL}}(t) = \int_0^\infty dz \langle L'(t) \bar{V}_{\text{RQL}}(t, t-z) L'(t-z) \rangle \bar{V}_{\text{RQL}}(t-z, t) \quad \text{II.29}$$

~~II.0. Kinetic equation for turbulent "collisionless" plasmas~~

Equation (II.20,21) describes the kinetic evolution of the distribution function $\bar{f} = \langle N \rangle$: a priori it involves all collisional effects. It has however the same apparent form as the equation obtained usually by introducing fluctuations

in the collisionless Vlasov equation, although such a procedure is not clearly justified usually. The equation obtained here is the actual justification of this usual procedure. Here the effect of individual particle collisions only appears later if one takes into account the specificity of the Klimontovich equation: this appears in the binary correlations

$$\bar{F}_{12} = \langle N_1(x_1, v_1, t) N_2(x_2, v_2, t) \rangle$$

which involves the "self correlations"

$$g_{12}^{\text{SELF}} = \delta(x_1 - x_2) \delta(v_1 - v_2) \langle N_1(x_1, v_1, t) \rangle \quad \text{II.30}$$

in addition to the usual ("distinct") correlations/19/ :

$$\bar{F}_{12} = \bar{F}_1 \bar{F}_2 + g_{12} + g_{12}^{\text{SELF}} \quad \text{II.31}$$

For practical purpose, it is sufficient to neglect self-correlations in order to describe the so-called "collisionless plasmas" from the Klimontovich equation. Correlations are then turbulent correlations, and the equations are identical to those obtained by the common procedure of the "fluctuating Vlasov equation" but the justification is more clearly exhibited.

II.E. Quasi-linear equation for "collisionless" turbulent plasmas

Approximating the general kinetic equation II.20 or II.21 consists in approximating the propagators. The quasi-linear approximation consists in retaining in the turbulent collisions II.24 only free trajectories (in the average field) determined by II.26. One obtains

$$\partial_t \bar{F}(t) = [\bar{L}(t) + G_{\text{QL}}(t)] \bar{F}(t) \quad 3.32$$

where

$$G_{\text{QL}}(t) = \int_0^\infty d\tau \langle L'(t) U^0(t, t-\tau) L'(t-\tau) \rangle U^0(t-\tau, t) \quad \text{II.33}$$

The free propagator U^0 is given by

$$\partial_t U^0(t, t_0) = \bar{L}(t) U^0(t, t_0) \quad \text{II.34}$$

i.e.

$$U^0(t, t_0) = X_+ \exp \int_{t_0}^t dt' [-\underline{v} \cdot \underline{\nabla} - \langle \underline{E}(x, t') \rangle \cdot \frac{\partial}{\partial \underline{v}}] \quad \text{II.35}$$

and, in the absence of any average field (i.e. for an homogeneous ensemble) it reduces to

$$U_0(t, t_0) = e^{-(t-t_0) \underline{v} \cdot \underline{\nabla}} \quad \text{II.36}$$

It is important to remark in the collision term II.33 that the propagator $U^0(t, \tau, t)$ takes into account (here in an approximative way) the non-Markovian feature of the general kinetic equation. The effect of the propagator $U^0(t, t, \tau)$ in the electric field correlation consists to introduce a Lagrangian correlation, here taken along a free motion (in a renormalized theory: along the average motion).

The explicit form of the quasi-linear equation can be easily obtained in the absence of an average field. We then have

$$\bar{L} = -\underline{v} \cdot \underline{\nabla}$$

$$U^0(t, t_0) \rightarrow U_0(t, t_0) = e^{-(t-t_0) \underline{v} \cdot \underline{\nabla}}$$

and

$$[\partial_t + \underline{v} \cdot \underline{\nabla}] \bar{f}(x, \underline{v}, t) = \frac{\partial}{\partial \underline{v}} \cdot \int_0^{\infty} d\tau \langle \underline{E}'(x, t) e^{-\tau \underline{v} \cdot \underline{\nabla}} \underline{E}(x, t-\tau) \rangle \cdot \frac{\partial}{\partial \underline{v}} e^{+\tau \underline{v} \cdot \underline{\nabla}} \bar{f}(x, \underline{v}, t) \quad \text{II.37}$$

Due to the obvious non-commutation

$$\frac{\partial}{\partial \underline{v}} e^{\tau \underline{v} \cdot \underline{\nabla}} = e^{\tau \underline{v} \cdot \underline{\nabla}} \left[\frac{\partial}{\partial \underline{v}} + \tau \underline{\nabla} \right] \quad \text{II.38}$$

a gradient term appears (beside the usual diffusion term in velocity space) which comes from the non-Markovian feature of the kinetic equation :

$$\left[\partial_t + \underline{v} \cdot \underline{\nabla} \right] \bar{F}(\underline{x}, \underline{v}, t) = \left[\frac{\partial}{\partial \underline{v}} \cdot \underline{D}(\underline{v}, t) \cdot \frac{\partial}{\partial \underline{v}} + \frac{\partial}{\partial \underline{v}} \cdot \underline{F}(\underline{v}, t) \cdot \underline{\nabla} \right] \bar{F}(\underline{x}, \underline{v}, t) \quad \text{II.39}$$

where

$$\underline{D}(\underline{v}, t) = \int_0^\infty d\tau \langle \underline{E}'(\underline{x}, t) \underline{E}'(\underline{x} - \tau \underline{v}, t - \tau) \rangle = \int_0^\infty d\tau \int d\underline{k} e^{i \underline{k} \cdot \underline{v} \tau} \underline{S}_{\underline{k}}^t(\tau) \quad \text{II.40}$$

and

$$\underline{F}(\underline{v}, t) = \int_0^\infty d\tau \tau \langle \underline{E}'(\underline{x}, t) \underline{E}'(\underline{x} - \tau \underline{v}, t - \tau) \rangle = \int_0^\infty d\tau \tau \int d\underline{k} e^{i \underline{k} \cdot \underline{v} \tau} \underline{S}_{\underline{k}}^t(\tau) \quad \text{II.41}$$

This last non-Markovian term has been shown to be of importance in presence of a strong magnetic field /19/. Here the fluctuation spectrum is given by the Eulerian correlation (assumed homogeneous here for simplicity):

$$\underline{S}_{\underline{k}}^t(\tau) = \int d\underline{r} e^{-i \underline{k} \cdot \underline{r}} \langle \underline{E}'(\underline{x}, t) \underline{E}'(\underline{x} - \underline{r}, t - \tau) \rangle \quad \text{II.42}$$

III. SOLUTION OF THE ASYMPTOTIC QUASI-LINEAR EQUATION : THE PROPAGATOR METHOD.

The solution of equation 1.39 is written in terms of the turbulent propagator \bar{V}^0

$$\bar{f}(x, y, t) = \bar{V}^0(t, t_0) \bar{f}(x, y, t_0) \quad \text{II.1}$$

which is defined by the same quasi-linear equation:

$$\partial_t \bar{V}^0(t, t_0) = \left[-y \cdot \nabla + \partial_y \cdot \underline{D}(y, t) \cdot \partial_y + \partial_y \cdot \underline{F}(y, t) \cdot \nabla \right] \bar{V}^0(t, t_0) \quad \text{II.2}$$

where $\partial_y = \frac{\partial}{\partial y}$ and $\nabla = \frac{\partial}{\partial x}$. The formal solution is a time-ordered exponential operator:

$$\bar{V}^0(t, t_0) = X_+ \exp \int_{t_0}^t dt' \left[-y \cdot \nabla + \partial_y \cdot \underline{D}(y, t') \cdot \partial_y + \partial_y \cdot \underline{F}(y, t') \cdot \nabla \right] \quad \text{III.3.}$$

where X_+ is the operator of time-ordering^{/16/} which prescribes the ordering of the different factors coming from the expansion of the exponential, in the order of decreasing times from left to right. Non-commutation of the different terms in the exponent can be taken into account by means of an interaction representation. In a general way one can show /16/ that the solution

$$U(t, t_0) = X_+ \exp \int_{t_0}^t dt' [A(t') + B(t')] \quad \text{II.4}$$

of the equation

$$\partial_t U(t, t_0) = [A(t) + B(t)] U(t, t_0) \quad \text{III.5}$$

(where **A** and **B** do not commute) is given in terms of the (assumed elementary) propagator

$$a \quad U_A(t, t_0) = A(t) U_A(t, t_0)$$

by the following formula (see eq.3.10 in Ref./16/) :

$$U(t, t_0) = U_A(t, t_0) X_+ \exp \int_{t_0}^t dt' U_A(t_0, t') B(t') U_A(t', t_0) \quad \text{II.7.}$$

By choosing $A = -\underline{v} \cdot \underline{\nabla}$ we obtain

$$\bar{V}^0(t, t_0) = U^0(t, t_0) X_+ \exp \int_{t_0}^t dt' U^0(t_0, t') \underline{\nabla} \cdot [\underline{D}(\underline{v}, t') \cdot \underline{\nabla} + \underline{F}(\underline{v}, t') \cdot \underline{\nabla}] U^0(t', t_0) \quad \text{III.8}$$

which can be put into the form

$$\bar{V}^0(t, t_0) = e^{-\tau \underline{v} \cdot \underline{\nabla}} X_+ \exp \int_0^\tau d\tau' (\partial_i - \tau' \nabla_i) \left[D_{ij}(\underline{v}, t - \tau + \tau') (\partial_j - \tau' \nabla_j) + F_{ij}(\underline{v}, t - \tau + \tau') \nabla_j \right] \quad \text{III.9.}$$

or

$$\bar{V}^0(t, t_0) = e^{-\tau \underline{v} \cdot \underline{\nabla}} X_+ e^{\beta_j(\tau, \underline{v}) \cdot \underline{\nabla} + \gamma_{ij}(\tau, \underline{v}) \nabla_j \partial_i + \delta_{ij}(\tau, \underline{v}) \nabla_i \nabla_j + \partial_i \alpha_{ij}(\tau, \underline{v}) \partial_j} \quad \text{III.10.}$$

The general form of the coefficients $\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}$ has been given in eqs (5.14-18) in Ref. /1/. In the simple case of a stationary spectrum, \underline{D} and \underline{F} do not depend on time. If we neglect in the exponent terms quadratic in the spectrum \underline{S} (i.e. the weak coupling approximation) the result can be written in the following simple and tractable form:

$$\bar{V}^0(t, t_0) = e^{-\tau \underline{v} \cdot \underline{\nabla}} e^{\beta_j(\tau, \underline{v}) \cdot \underline{\nabla}} e^{\gamma_{ij}(\tau, \underline{v}) \nabla_j \partial_i} e^{\delta_{ij}(\tau, \underline{v}) \nabla_i \nabla_j} e^{\partial_i \alpha_{ij}(\tau, \underline{v}) \partial_j} \quad \text{III.11}$$

where

$$\alpha_{ij}(\tau, \underline{v}) = \tau D_{ij}(\underline{v}) \quad \text{III.12}$$

$$\beta_j(\tau, \underline{v}) = -\frac{\tau^2}{2} \frac{\partial D_{ij}(\underline{v})}{\partial v_i} + \tau \frac{\partial F_{ij}}{\partial v_i} \quad \text{III.13}$$

$$\gamma_{ij}(\tau, \underline{v}) = -\frac{\tau^2}{2} (D_{ij}(\underline{v}) + D_{ji}(\underline{v})) + \tau F_{ij}(\underline{v}) \quad \text{III.14}$$

$$\delta_{ij}(\tau, \underline{v}) = \frac{\tau^3}{3} D_{ij}(\underline{v}) - \frac{\tau^2}{2} F_{ij}(\underline{v}) \quad \text{III.15}$$

The physical meaning of these various coefficients $\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\delta}$ may be obtained in the following way. By integrating the exact equations of motion

$$\begin{cases} \dot{\underline{x}}(t) = \underline{v}(t) \\ \dot{\underline{v}}(t) = \underline{E}[\underline{x}(t), t] \end{cases} \quad .10$$

up to the order $E^2 \omega^2$ and by taking the long time limit, we obtain:

$$\langle \underline{x}(t-\tau) \rangle = \underline{x} - \tau \underline{v} + \underline{\beta}(\underline{v}, \tau) \quad \text{III.17}$$

$$\langle \underline{v}_j(t-\tau) \rangle = \underline{v}_j + \frac{\partial \alpha_{ij}(\underline{v}, \tau)}{\partial v_i} \quad \text{IV.18}$$

$$\langle v'_j(t-\tau) v'_n(t-\tau) \rangle = \alpha_{jn}(\underline{v}, \tau) + \alpha_{nj}(\underline{v}, \tau) \quad \text{IV.19}$$

$$\langle x'_j(t-\tau) v'_n(t-\tau) \rangle = \gamma_{nj}(\underline{v}, \tau) \quad \text{IV.20}$$

$$\langle x'_j(t-\tau) x'_n(t-\tau) \rangle = \delta_{jn}(\underline{v}, \tau) + \delta_{nj}(\underline{v}, \tau) \quad \text{IV.21}$$

We see that $\underline{\beta}$ represents the turbulent contribution to the average displacement, the velocity dependence of the coefficient

$\underline{\alpha}$ represents an average acceleration or slowing down : these two terms are due to the velocity dependence of the diffusion coefficient. The term $\underline{\alpha}$ represents the elementary process of the quasi-linear description: diffusion in velocity space .

The term $\underline{\delta}$ represents the associated spatial diffusion ($\langle x'^2 \rangle \sim \tau^3$

the Dupree damping term) and the term $\underline{\gamma}$ represents an often neglected correlation between position and velocity. This whole set of terms have been discussed for the first time in Ref. /3 /.

Inversely one can check that the propagator obtained in III.11 allows one to recover results III.17-21, coming from the equations of motion, by letting the obtained propagator acting on $\underline{x} = \underline{x}(t)$ and $\underline{v} = \underline{v}(t)$ for instance:

$$\langle \underline{x}(t, \tau) \rangle = \bar{V}^0(t, t-\tau) \underline{x} = \underline{x} + \tau \underline{v} + \beta(\tau, \underline{v})$$

$$\langle \underline{v}(t, \tau) \rangle = \bar{V}^0(t, t-\tau) \underline{v} = \underline{v} + \frac{\tau \alpha_i(\tau, \underline{v})}{\partial v_i}$$

a.s.o.

The obtained propagator thus takes correctly into account (and allows to describe) turbulent trajectories in the quasi-linear, approximation, i.e. linearly in the spectrum.

In this weak-coupling approximation and for a stationary spectrum, we obtain the solution of the quasi-linear equation in terms of the initial value $\bar{f}(\underline{x}, \underline{v}, 0)$ as

$$\bar{f}(\underline{x}, \underline{v}, \tau) = \bar{V}^0(\tau, 0) \bar{f}(\underline{x}, \underline{v}, 0) \\ = e^{-\tau \underline{v} \cdot \underline{\nabla}} e^{\tau \beta \cdot \underline{\nabla}} e^{\tau \gamma_{ij} \nabla_j \partial_i} e^{\tau \delta_{ij} v_i \nabla_j} e^{\tau \alpha_i \partial_i \nabla_j \partial_j} \bar{f}(\underline{x}, \underline{v}, 0) \quad \text{III.22}$$

Let us introduce the Fourier transform by

$$\bar{f}(\underline{x}, \underline{v}) = \int d\underline{k} e^{i\underline{k} \cdot \underline{x}} \bar{f}_{\underline{k}}(\underline{v}) \quad \text{III.23}$$

$$\bar{f}_{\underline{k}}(\underline{v}) = \frac{1}{8\pi^3} \int d\underline{x} e^{-i\underline{k} \cdot \underline{x}} \bar{f}(\underline{x}, \underline{v}) \quad \text{III.24}$$

We thus have

$$\bar{f}_{\underline{k}}(\underline{v}, \tau) = \bar{V}_{\underline{v} = i\underline{k}}^0(\tau, 0) \bar{f}_{\underline{k}}(\underline{v}, 0) \quad \text{III.25}$$

or

$$\bar{f}_{\underline{k}}(\underline{v}, \tau) = e^{-i\underline{k} \cdot \underline{v} \tau} e^{i\underline{k} \cdot \beta(\underline{v}, \tau)} e^{i\underline{k}_j \gamma_{ij}(\underline{v}, \tau) \partial_i} e^{-i\underline{k} \cdot \underline{\delta}(\underline{v}, \tau) \cdot \underline{k}} e^{\tau \alpha_i(\underline{v}, \tau) \partial_i} \bar{f}_{\underline{k}}(\underline{v}, 0)$$

III.26

For later use, let us also introduce the Laplace transform

$$F(\omega) = \int_0^{\infty} dt e^{i\omega t} f(t)$$

We obtain

$$\bar{F}_k(\underline{v}, \omega) = \int_0^{\infty} d\tau e^{i(\omega - \underline{k} \cdot \underline{v})\tau} e^{i\underline{k} \cdot \underline{\beta}(\underline{v}, \tau)} e^{i\underline{k}_j \gamma_{ij}(\underline{v}, \tau) \partial_i} e^{-\underline{k} \cdot \underline{\delta}(\underline{v}, \tau) \underline{k}} e^{\underline{\partial} \cdot \underline{\alpha}(\underline{v}, \tau) \underline{\partial}} \bar{F}_k(\underline{v}, t=0) \quad \text{III.27}$$

In the simple case of a stationary spectrum the solution of II.2 can be written

$$\bar{V}^0(t, 0) = e^{-t \underline{v} \cdot \underline{\nabla} + t \underline{\partial} \cdot [\underline{D}(\underline{v}) \cdot \underline{\partial} + \underline{F}(\underline{v}) \cdot \underline{\nabla}]} \quad \text{IV.28}$$

and the solution III.27 can also be written formally as

$$\begin{aligned} \bar{F}_k(\underline{v}, \omega) &= \int_0^{\infty} d\tau e^{i\omega\tau} \bar{V}_{\underline{v}=\underline{ik}}^0(\tau, 0) \bar{F}_k(\underline{v}, 0) \\ &= \int_0^{\infty} d\tau e^{i\omega\tau} e^{-i\underline{k} \cdot \underline{v} \tau + \tau \underline{\partial} \cdot [\underline{D}(\underline{v}) \cdot \underline{\partial} + i \underline{F}(\underline{v}) \cdot \underline{k}]} \bar{F}_k(\underline{v}, t=0) \quad \text{III.29.} \end{aligned}$$

that will be compared later to the non-asymptotic case.

IV. RENORMALIZED QUASI-LINEAR KINETIC EQUATION

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Renormalizing the quasi-linear equation essentially consists in taking into account the turbulent propagator in the collision term, instead of the free propagator. One thus have (see II.21):

$$\partial_t \bar{f}(t) = [\bar{L}(t) + G(t)] \bar{f}(t) \quad \text{IV.1}$$

with

$$G(t) = \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \quad \text{IV.2}$$

and

$$\bar{V}(t, t_0) = U^0(t, t_0) + \int_{t_0}^t dt' U^0(t, t') \int_0^\infty d\tau \langle L'(t') \bar{V}(t', t'-\tau) L'(t'-\tau) \rangle \bar{V}(t'-\tau, t') \bar{V}(t', t_0) \quad \text{IV.3}$$

This last equation is the RQL or DIA approximation of the DYSON equation II.27. It is equivalent to approximate the Weinstock propagator Λ by the turbulent propagator \bar{V} . In equation II.16 the present approximation is equivalent to keeping only the first term in the r.h.s., (next term are thus responsible for "clumps" and other higher non-linear effects).

The weak-coupling approximation of eq. q.2 for the propagator consists in considering in this equation only the quasi-linear approximation of the collision term:

$$\bar{V}(t, t_0) = U^0(t, t_0) + \int_{t_0}^t dt' U^0(t, t') \int_0^\infty d\tau \langle L'(t') U^0(t', t'-\tau) L'(t'-\tau) \rangle U^0(t'-\tau, t') \bar{V}(t', t_0) \quad \text{z9}$$

This is the BOURRET approximation of the DYSON equation /17/.

It is equivalent to

$$\partial_t \bar{V}(t, t_0) = [\bar{L}(t) + G_{QL}(t)] \bar{V}(t, t_0) \quad \text{IV.5}$$

and the solution

$$\bar{V}(t, t_0) = X_+ \exp \int_{t_0}^t dt' \left[L(t') + \partial \cdot \underline{D}(v, t') \cdot \partial + \partial \cdot \underline{F}(v, t') \cdot \nabla \right]$$

has been developed explicitly in II.10-11 ,

The important physical effects introduced by this renormalized approximation is the main motivation of the present comparison between the techniques of propagator and Green's functions.

V. NON-ASYMPTOTIC QUASI-LINEAR EQUATION

The asymptotic form of the general kinetic equation II.20 or of the quasi-linear one II.32 has been obtained either by the projector technique and the subdynamics method, or by the usual arguments:

$$\lim_{t \rightarrow t_0} F'(t_0) \neq 0 \quad \text{V.A}$$

Some authors however consider the non-asymptotic equation obtained by simply neglecting the initial fluctuation in the general master equation. In spite of the fact that there do not exist, to the best of our knowledge, any rigorous justification of the consistency of such an equation, one can consider that the long 'time limit may be taken afterwards; such an equation presents some advantage namely for introducing Laplace transforms.

In the master equation II.19 we simply take

$$\begin{cases} t_0 = 0 \\ F'(0) = 0 \end{cases} \quad \text{V.2}$$

and limit ourselves to the quasi-linear approximation

$$\Lambda(t, t') \rightarrow U^0(t, t') \quad \text{V.3.}$$

We then obtain the non-asymptotic quasi-linear equation (E) $= 0$

$$\begin{aligned} [\partial_t + \underline{v} \cdot \underline{\nabla}] \bar{F}(\underline{x}, \underline{v}, t) &= \frac{\partial}{\partial \underline{v}} \cdot \int_0^t d\tau \langle \underline{E}'(\underline{x}, t) \underline{E}'(\underline{x} - \underline{v}\tau, t - \tau) \rangle e^{-\tau \underline{v} \cdot \underline{\nabla}} \bar{F}(\underline{x}, \underline{v}, t - \tau) \\ &= \int_0^t d\tau K'(\tau) \bar{F}(\underline{x}, \underline{v}, t - \tau) \end{aligned} \quad \text{V.4.}$$

where the integral kernel can be written in terms of the (Eulerian) spectrum :

$$K'(\tau) = \frac{\partial}{\partial \underline{v}} \cdot \int d\underline{\ell} e^{i \underline{\ell} \cdot \underline{v} \tau} \underline{S}_{\underline{\ell}}(\tau) e^{-\tau \underline{v} \cdot \underline{\nabla}} \cdot \frac{\partial}{\partial \underline{v}}$$

We assume here an homogeneous spectrum (\underline{S} is \underline{x} -independent).

In Fourier transform (III.23) we have

$$[\partial_t + i \underline{k} \cdot \underline{v}] \bar{F}_{\underline{k}}(\underline{v}, t) = \frac{1}{8\pi^3} \int d\underline{x} e^{-i \underline{k} \cdot \underline{x}} \int_0^t d\tau \frac{\partial}{\partial \underline{v}} \cdot \int d\underline{\ell} e^{i \underline{\ell} \cdot \underline{v} \tau} \sum_{\underline{\ell}}^t(\tau) e^{-i \underline{\ell} \cdot \underline{v} \tau} \frac{\partial}{\partial \underline{v}} \bar{F}(\underline{x}, \underline{v}, t-\tau) \quad \text{V.6}$$

or

$$[\partial_t + i \underline{k} \cdot \underline{v}] \bar{F}_{\underline{k}}(\underline{v}, t) = \int_0^t d\tau K(\tau) \bar{F}_{\underline{k}}(t-\tau) \equiv \frac{\partial}{\partial \underline{v}} \cdot \int_0^t d\tau H(\underline{k}, \underline{v}, \tau) \cdot \frac{\partial}{\partial \underline{v}} \bar{F}_{\underline{k}}(\underline{v}, t-\tau) \quad \text{V.7.}$$

where

$$K(\tau) = \frac{\partial}{\partial \underline{v}} \cdot \int d\underline{\ell} e^{i(\underline{\ell} - \underline{k}) \cdot \underline{v} \tau} \sum_{\underline{\ell}}^t(\tau) \cdot \frac{\partial}{\partial \underline{v}} \equiv \frac{\partial}{\partial \underline{v}} \cdot H(\underline{k}, \underline{v}, \tau) \cdot \frac{\partial}{\partial \underline{v}} \quad \text{V.8.}$$

The so-obtained equation (V.7) exhibits both non-Markovian ($\bar{F}(t-\tau) \neq \bar{F}(t)$) and non-asymptotic ($t \neq \infty$) features. By neglecting these two effects, the turbulent collision term would reduce to

$$\frac{\partial}{\partial \underline{v}} \cdot \int_0^\infty d\tau H(\underline{k}, \underline{v}, \tau) \cdot \frac{\partial}{\partial \underline{v}} \quad \text{V.9.}$$

where

$$\int_0^\infty d\tau H(\underline{k} = 0, \underline{v}, \tau) = \int_0^\infty d\tau \int d\underline{\ell} e^{i \underline{\ell} \cdot \underline{v} \tau} \sum_{\underline{\ell}}^t(\tau) = \underline{D}(\underline{v}, t) \quad \text{V.10.}$$

gives the usual velocity-diffusion term in the homogeneous case ($\underline{k} = 0$).

Let us introduce the Laplace transform of the complete equation (V.7) :

$$F(\omega) = \int_0^\infty dt e^{i \omega t} F(t) \quad \text{I.11}$$

and let us restrict ourselves to the case of a time-independent spectrum. Equation V.7 then writes:

$$[-i\omega + i\mathbf{k} \cdot \mathbf{v}] \bar{f}_{\mathbf{k}}(\mathbf{v}, \omega) = K(\omega) \bar{f}_{\mathbf{k}}(\mathbf{v}, \omega) + \bar{f}_{\mathbf{k}}(\mathbf{v}, t=0) \quad \text{v.12.}$$

where

$$K(\omega) = \int_0^\infty d\tau e^{i\omega\tau} K(\tau) = \frac{\partial}{\partial \mathbf{v}} \cdot \underline{\underline{H}}(\mathbf{k}, \mathbf{v}, \omega) \cdot \frac{\partial}{\partial \mathbf{v}} \quad \text{v.13}$$

and

$$\underline{\underline{H}}(\mathbf{k}, \mathbf{v}, \omega) = \int_0^\infty d\tau e^{i(\omega - \mathbf{k} \cdot \mathbf{v})\tau} \int d\mathbf{l} e^{i\mathbf{l} \cdot \mathbf{v}\tau} \underline{\underline{S}}_{\mathbf{l}}(\tau) = \underline{\underline{H}}(\omega - \mathbf{k} \cdot \mathbf{v}, \mathbf{v}) \quad \text{v.14}$$

such that

$$\underline{\underline{H}}(\omega - \mathbf{k} \cdot \mathbf{v}, \mathbf{v}) = \underline{\underline{D}}(\mathbf{v}) \quad \text{v.15}$$

VI. SOLUTION OF THE NON-ASYMPTOTIC QUASI-LINEAR EQUATION:

GREEN'S FUNCTION

VI.A. The Green function given by HORTON

By writing equation 4.12 in the form

$$-i [\omega - \underline{k} \cdot \underline{v} - i k(\omega)] \bar{F}_{\underline{k}}(\underline{v}, \omega) = \bar{F}_{\underline{k}}(\underline{v}, t=0) \quad \text{VI.1}$$

one can define the associated Green function by the equation

$$-i [\omega - \underline{k} \cdot \underline{v} - i k(\omega)] G_{\omega, \underline{k}}(\underline{v}, \underline{v}') = \delta(\underline{v} - \underline{v}') \quad \text{VI.2.}$$

and the solution of 4.16 can be written

$$\bar{F}_{\underline{k}}(\underline{v}, \omega) = \int d\underline{v}' G_{\omega, \underline{k}}(\underline{v}, \underline{v}') \bar{F}_{\underline{k}}(\underline{v}', t=0) \quad \text{VI.3.}$$

The solution of equation VI.2 has been given by HORTON et al. / 5 / in an approximation which is equivalent to the weak-coupling renormalized approximation used in Section III, in the propagator formalism for the asymptotic equation. Our aim is to compare the approximations used in the two formalisms. Their solution writes [eq. 6.12 in Ref. /5/]:

$$G_{\omega, \underline{k}}(\underline{v}, \underline{v}') = \int_0^\infty d\tau \frac{1}{(4\pi H \tau)^{3/2}} e^{-\frac{k^2 H \tau^3}{12}} e^{-i [\omega - \frac{1}{2}(\underline{k} \cdot \underline{v} + \underline{k} \cdot \underline{v}')] \tau} e^{-\frac{(\underline{v} - \underline{v}')^2}{4 H \tau}} \quad \text{VI.4.}$$

This form actually assumes an isotropic tensor

$$\underline{v} \cdot \underline{H} \cdot \underline{v} = H v^2$$

$$\underline{k} \cdot \underline{H} \cdot \underline{k} = H k^2$$

VI.5

This is equivalent to neglecting the velocity dependence $\frac{\partial H}{\partial \underline{v}} = 0$ as well for the dependence of \underline{H} in $\omega - \underline{k} \cdot \underline{v}$ as for its dependence in \underline{v} .

In the framework of these approximations, we will compare in Section@ with the asymptotic solution obtained from the propagator 8.26. .

Let us introduce a Fourier transform in velocity space:

$$\begin{aligned}\bar{F}_{\underline{k}}(\underline{v}') &= \int d\underline{q} e^{i\underline{q} \cdot \underline{v}'} \bar{F}_{\underline{k}, \underline{q}} \\ \bar{F}_{\underline{k}, \underline{q}} &= \frac{1}{8\pi^3} \int d\underline{v}' e^{-i\underline{q} \cdot \underline{v}'} \bar{F}_{\underline{k}}(\underline{v}')\end{aligned}\quad \text{VI.6.}$$

The solution VI.3 can thus be written in the considered approximation:

$$\begin{aligned}\bar{F}_{\underline{k}}(\underline{v}, \omega) &= \int d\underline{v}' \int_0^\infty d\tau \frac{1}{(4\pi H\tau)^{3/2}} e^{-\frac{k^2 H\tau^3}{12}} e^{i[\omega - \frac{1}{2}(\underline{k} \cdot \underline{v} + \underline{k} \cdot \underline{v}')] \tau} \\ &\quad e^{-\frac{(\underline{v} - \underline{v}')^2}{4H\tau}} \int d\underline{q} e^{i\underline{q} \cdot \underline{v}'} \bar{F}_{\underline{k}, \underline{q}}(t=0)\end{aligned}\quad \text{VI.7}$$

The $d\underline{v}'$ integration can be performed ($\underline{v}' = \underline{v} - \underline{w}$)

$$\begin{aligned}\int d\underline{v}' e^{-\frac{i}{2} \underline{k} \cdot \underline{v}'} e^{i\underline{q} \cdot \underline{v}'} e^{-\frac{(\underline{v} - \underline{v}')^2}{4H\tau}} &= e^{i\underline{v} \cdot (\underline{q} - \frac{1}{2} \underline{k} \tau)} \int d\underline{w} e^{-i\underline{w} \cdot [\underline{q} - \frac{1}{2} \underline{k} \tau]} e^{-\frac{\underline{w}^2}{4H\tau}} \\ &= (4\pi H\tau)^{3/2} e^{i\underline{v} \cdot (\underline{q} - \frac{1}{2} \underline{k} \tau)} e^{-(\underline{q} - \frac{1}{2} \underline{k} \tau)^2 H\tau}\end{aligned}\quad \text{VI.8}$$

by using

$$\int d\underline{w} e^{\underline{q} \cdot \underline{w}} e^{-b\underline{w}^2} = \frac{\pi^{3/2}}{b^{3/2}} e^{q^2/4b} \quad \text{VI.9}$$

The solution VI.7 can thus be written

$$\bar{F}_{\underline{k}}(\underline{v}, \omega) = \int d\underline{q} \bar{F}_{\underline{k}, \underline{q}}(t=0) \int_0^\infty d\tau e^{-\frac{k^2 H\tau^3}{12}} e^{i\omega\tau} e^{-\frac{i}{2} \underline{k} \cdot \underline{v} \tau} e^{i\underline{v} \cdot (\underline{q} - \frac{1}{2} \underline{k} \tau)} e^{-(\underline{q} - \frac{1}{2} \underline{k} \tau)^2 H\tau} \quad \text{VI.10.}$$

By collecting terms $H\tau^3$ we finally have:

$$\bar{F}_{\underline{k}}(\underline{v}, \omega) = \int d\underline{q} \bar{F}_{\underline{k}, \underline{q}}(t=0) \int_0^\infty d\tau e^{i\underline{q} \cdot \underline{v} \tau} e^{i\omega \tau} e^{-i\underline{k} \cdot \underline{v} \tau} e^{-\frac{\tau^3}{3} k^2 H} e^{-\tau q^2 H} e^{+\tau^2 \underline{q} H \underline{k}} \quad \text{VI.11}$$

One recognizes here, in a scalar form, the spatial diffusion term $-\frac{1}{3}\tau^3 k^2 H$ (Dupree damping), the velocity diffusion term ($q^2 H$) and also the crossed term $\underline{q} H \underline{k}$ corresponding to the correlations between position and velocity.

VI.B The propagator formalism

Solution of eq. VI.11 can be written

$$\begin{aligned} \bar{F}_{\underline{k}}(\underline{v}, \omega) &= [-i(\omega - \underline{k} \cdot \underline{v} - iK(\omega))]^{-1} \bar{F}_{\underline{k}}(\underline{v}, t=0) \\ &= \int_0^\infty d\tau e^{i[\omega - \underline{k} \cdot \underline{v} - iK(\omega)]\tau} \bar{F}_{\underline{k}}(\underline{v}, t=0) \\ &= \int_0^\infty d\tau e^{i\omega \tau} \left\{ e^{-i\underline{k} \cdot \underline{v} \tau + \tau \frac{\partial}{\partial \underline{v}} \cdot \underline{H}(\omega - \underline{k} \cdot \underline{v}, \underline{v}) \cdot \frac{\partial}{\partial \underline{v}}} \right\} \bar{F}_{\underline{k}}(\underline{v}, t=0) \quad \text{VI.12} \end{aligned}$$

Here a propagator appears

$$\bar{U}_{\omega - \underline{k} \cdot \underline{v}}(\tau, 0) = e^{-i\underline{k} \cdot \underline{v} \tau + \tau \frac{\partial}{\partial \underline{v}} \cdot \underline{H}(\omega - \underline{k} \cdot \underline{v}, \underline{v}) \cdot \frac{\partial}{\partial \underline{v}}} \quad \text{VI.13}$$

which is solution of

$$\frac{\partial}{\partial \tau} \bar{U}_{\omega - \underline{k} \cdot \underline{v}}(\tau, 0) = \left[-i\underline{k} \cdot \underline{v} + \frac{\partial}{\partial \underline{v}} \cdot \underline{H}(\omega - \underline{k} \cdot \underline{v}, \underline{v}) \cdot \frac{\partial}{\partial \underline{v}} \right] \bar{U}_{\omega - \underline{k} \cdot \underline{v}}(\tau, 0) \quad \text{VI.14}$$

and it generalizes to the non-asymptotic case the previous propagator (for stationary spectrum) :

$$\bar{V}_{\underline{v}=ik}(\tau, 0) = e^{-i \underline{k} \cdot \underline{v} \tau + \tau \frac{\partial}{\partial \underline{v}} \cdot \left[\underline{D}(\underline{v}) \cdot \frac{\partial}{\partial \underline{v}} + \underline{F}(\underline{v}) \cdot \underline{v} \right]} \quad \text{VI.15}$$

It can be seen that in the non-asymptotic case (VI.14) the non-Markovian contributions are taken into account in a much simpler way as compared to the asymptotic case (F terms in VI.15). This is due to the tensor $\underline{H}(\omega, \underline{k}, \underline{v}, \underline{v})$: its approximate form for $\omega = \underline{k} \cdot \underline{v}$ is equivalent to the Markovian approximation ($\underline{F} = 0$) of the asymptotic case because

$$\underline{H}(\omega = \underline{k} \cdot \underline{v}, \underline{v}) = \underline{D}(\underline{v}) \quad \text{VI.16}$$

The knowledge of the expansion of the non-commuting operators in the exponent of (VI.15) :

$$\bar{V}_{\underline{v}=ik}(\tau, 0) = e^{-i \underline{k} \cdot \underline{v} \tau} e^{i \underline{k} \cdot \underline{v} \tau} e^{i \underline{k}_j \gamma_{ij}} = e^{-\underline{k} \cdot \underline{D} \cdot \underline{k}} e^{\underline{v} \cdot \underline{D} \cdot \underline{v}} \quad \text{VI.17}$$

which for $\underline{F} = 0 = \frac{\partial S}{\partial t}$ gives

$$\bar{V}_{\underline{v}=ik}(\tau, 0) = e^{-i \underline{k} \cdot \underline{v} \tau} e^{-i \frac{\tau^2}{2} \underline{k}_j \frac{\partial D_{ij}}{\partial v_i}} e^{-i \frac{\tau^2}{2} \underline{k}_j (D_{ij} + D_{ji}) \partial_i} e^{-\frac{\tau^3}{3} \underline{k} \cdot \underline{D} \cdot \underline{k}} e^{\tau \underline{v} \cdot \underline{D} \cdot \underline{v}} \quad \text{VI.18}$$

we immediately obtain here in the same approximation:

$$\bar{U}_{\omega, \underline{k}, \underline{v}}(\tau, 0) = e^{-i \underline{k} \cdot \underline{v} \tau} e^{-i \frac{\tau^2}{2} \frac{\partial H^{ij}(\omega, \underline{k}, \underline{v}, \underline{v})}{\partial v_i}} e^{-i \frac{\tau^2}{2} \underline{k}_j (H_{ij} + H_{ji}) \partial_i} e^{-\frac{\tau^3}{3} \underline{k} \cdot \underline{H} \cdot \underline{k}} e^{\tau \underline{v} \cdot \underline{H} \cdot \underline{v}} \quad \text{VI.19}$$

The solution VI.12 can thus be written as

$$\bar{F}_{\underline{k}}(\underline{v}, \omega) = \int_0^{\infty} d\tau e^{-i(\omega - \underline{k} \cdot \underline{v})\tau} e^{-i \frac{\tau^2}{2} \frac{\partial H^{ij}}{\partial v_i}} e^{-i \frac{\tau^2}{2} \underline{k}_j (H_{ij} + H_{ji}) \partial_i} e^{-\frac{\tau^3}{3} \underline{k} \cdot \underline{H} \cdot \underline{k}} e^{\tau \underline{v} \cdot \underline{H} \cdot \underline{v}} \bar{F}_{\underline{k}}(\underline{v}, t=0) \quad \text{VI.20}$$

VI.C. Comparison =====

When compared with the expression VI.11 obtained from the Green function, this last result has the advantage of taking into account the dependence of \underline{H} upon both $\omega - \underline{k} \cdot \underline{v}$ and \underline{v} . The Fourier transform in velocity space, which has been introduced to integrate the Green function actually needed to neglect such a dependence. For this reason, one can say that the simple result obtained from the Green function from the non-asymptotic equation is equivalent to the approximation

$$\underline{H}(\omega - \underline{k} \cdot \underline{v}, \underline{v}) \rightarrow \underline{D}$$

performed in the result of the asymptotic equation if we further neglect the non-Markovian contributions (i.e. Eq. III.26 where $\underline{k}, \underline{v}, \underline{x}, \underline{\delta}$ are given by III.12-15 with $\underline{F} = 0$).

In conclusion, we have shown that the approximation used by HORTON et al. to calculate the Green function VI.4 in the case of the non-asymptotic quasi-linear equation is actually the same physical weak-coupling approximation used in the propagator formalism. However, the use of Green's functions seems to be more delicate if we would like to take into account

- the tensorial feature of \underline{H}
- but mainly its velocity-dependence (and thus the non-Markovian and non-asymptotic features).

Neglecting these two effects is equivalent to only consider the asymptotic and Markovian quasi-linear equation. On the other hand, the propagator technique allows one to take into account

- the tensorial feature of \underline{D} and \underline{F} in the asymptotic case (or of \underline{H} in the non-asymptotic case)
- their velocity dependence (i.e. average displacements effects: see II.17-18) and non-asymptotic contributions in the case of the tensor \underline{H}

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SECTION 4

A NEW KINETIC DESCRIPTION FOR TURBULENT COLLISION INCLUDING MODE-COUPLING

J. H. MISGUICH and C. M. TCHEN

ABSTRACT

The micro-dynamical state of fluid turbulence is described by a hydrodynamical system. This is transformed into a master equation in a form analogous to the Vlasov equation for plasma turbulence. When the total distribution function is decomposed into a mean value and a fluctuation, the evolution of the mean distribution satisfies a transport equation, called the kinetic equation, and contains a turbulent collision that represents the statistical effect of the turbulent fluctuations, while the evolution of the fluctuation will form a transport equation for the turbulent collision. The mechanism of this collision is investigated. The report compares our theory with the clump theory of Dupree, the renormalization theory of Misguich and Balescu and the direct interaction theory of Kraichnan.

The paper is completed by Dr. Tchen at the City College of New York in collaboration with Dr. Misguich at the Centre d'Etudes Nucleaires, Fontenay-aux-Roses, France.

The work is the result of cooperation between Dr. Tchen at the City College of New York and Dr. Misguich at the Centre d'Etudes Nucleaires, Fontenay-aux-Roses, France.

INTRODUCTION

In plasma and fluid turbulence, the main average effect of the fluctuations consists to enhance (or replace) the role of particle collisions and to enhance consequently the transport processes (turbulent diffusion, turbulent viscosity, α, β, \dots). The origin of such phenomena is generally referred to as "turbulent collisions".

The starting point of theoretical descriptions are the Navier-Stokes equation for the macroscopic description of fluids, and the Klimontovich equation for the microscopic description of plasmas. The latter reduces to the Vlasov equation in the case of "collisionless" plasmas, which behave like an incompressible fluid in phase space.

The most widely used among the theoretical descriptions of such processes are the Direct Interaction Approximation (DIA) introduced by Kraichnan in fluid turbulence, and the Renormalized Quasi-Linear (RQL) approximation introduced by Dupree and Weinstock in plasma turbulence. Both approximations have been shown to be analogous. /1/

Similar formalisms can thus be used to describe both kind of turbulence, and we will adopt here the language of plasma physics. The results can immediately be translated in the fluid language by replacing electric field fluctuations by pressure gradient fluctuations

Turbulent collisions effects are described in RQL or DIA approximations by means of a Lagrangian correlation of fluctuating fields between two points which are separated in space and in time. These distances in space and in time are actually related to each other by a complicated trajectory which involves all the dynamical problem; in RQL or DIA this trajectory reduces to an average diffusive trajectory,

The purpose of the present work is to analyse the effects of higher mode coupling terms on this approximate description. Dupree for instance has introduced so-called 'beta terms' in plasma turbulence, the role of which is to ensure energy conservation to a higher degree of precision. Such terms have been proved to be important in drift wave turbulence /2/.

The introduction of such higher order corrections has however been made in a rather intuitive manner : a simplified approach is presented in Section 11. Here we derive new results for turbulent collisions by using a general kinetic formulation of plasma turbulence (Section III). Moreover we prove that these higher order corrections already appears from the non-linear dynamics, even when the non-linearity introduced by Poisson equation is not yet taken into account: they cannot thus be reduced to self-consistency effects only.

Our main result (Eq. 111.14) consists to describe the deviation of the turbulent collision term from its RQL or DIA approximation in an exact way. This deviation involves a generalisation of the Beta terms introduced by Dupree, which includes here all higher order mode-coupling terms. A new approximation is proposed in Eq. (111.20) which goes beyond the RQL description; an approximate treatment of this equation is left for a future work.

Another approach is presented in Section IV, which uses a simple non kinetic treatment of the Vlasov equation. Although the resulting expression involves four main contributions (Eq. IV.24), we have been able to prove that in the kinetic regime an unexpected cancellation occurs between the last two contributions, and we recover the general result of the kinetic formulation.

The consequences of this cancellation are examined in Section V where we focus our attention on a quantity C_L defined as the infinite time integral of the Lagrangian autocorrelation of fluctuating fields. Exactly like in the classical case of Brownian motion, we demonstrate here that

in an asymptotic description, C_L goes to zero for an homogeneous and stationary turbulence. Of course, when different scales can be introduced to describe the fluctuations, the Lagrangian correlation of the small-scale fluctuations remain inhomogeneous and non-stationary, and this avoids the corresponding C_L^0 to vanish in the asymptotic case. In such a scaling description of turbulence, C_L^0 can be used as a bare description of the turbulent collision : higher order mode-coupling terms then will introduce a dynamical shielding effect. This description will be developed in a subsequent work where an explicit calculation of the intrinsic or shielded turbulent collision will be performed.

Solution of (II.3) can be obtained as

$$F'(t) = \int_{t_0}^t dt' U_0(t, t') [L'(t') \bar{F}(t') + BL'(t') F'(t')] \quad \text{II.7a}$$

or

$$F' = U_0 * [L' \bar{F} + BL' F'] \quad \text{II.7b}$$

where time dependence is taken into account by the notation $*$ which denotes the time convolution. This solution is formal since it is an integral equation which remains to be solved.

Substitution of (II.7) into (II.3) is equivalent to an iteration and gives

$$\partial_t F' = \bar{L} F' + L' \bar{F} + BL' U_0 * L' \bar{F} + BL' U_0 * BL' F' \quad \text{II.3b}$$

A simple handwaving argument can be applied in the last term in the r.h.s. : in $BL' F' = L' F' - \langle L' F' \rangle$ the projector B means that the average has to be subtracted from $L' F'$. In the final calculation of average quantities the L' cannot be taken in average with F' : simple averages involving this last L' can only be taken:

i) either with the other L' which yields

$$\langle L' U_0 * L' \rangle F'$$

ii) either with other fluctuating quantities which will appear to the right. This case can be schematized by an arrow

$$\langle L' U_0 * L' F' \rangle$$

which indicates that L' is operated by U_0 but is excluded from the bracket average.

The last term in (II.3b) thus involves at least the above two terms, and (II.3b) can be written:

$$\partial_t F' = \bar{L} F' + L' \bar{F} + BL' U_0 * L' \bar{F} + \underbrace{\langle L' U_0 * L' \rangle F'} + \underbrace{\langle L' U_0 * L' F' \rangle} \quad \text{II.8}$$

Both underlined terms can be used to define an average renormalized propagator \bar{V}^0 and the solution writes.

$$F' = \bar{V}^0 * [L' \bar{F} + BL' U_0 * L' \bar{F} + \underbrace{\langle L' U_0 * L' F' \rangle}] \quad \text{II.9}$$

When substituted into (II.2) this result allows us to write the average equation

$$\partial_t \bar{F} = \bar{L} \bar{F} + \langle L' \bar{V}^0 * L' \rangle \bar{F} + \langle L' \bar{V}^0 * B L' U_0 * L' \rangle \bar{F} \\ + \underbrace{\langle L' \bar{V}^0 * L' U_0 * L' F' \rangle}_{11.10}$$

In the r.h.s. the first term describes free motion in the average field, the second term a turbulent collision term (here in the weak-coupling RQL approximation). The third and 4th terms are higher order corrections to this approximation of the turbulent collisions. The complete expression for the collision appears to be in this handwaving derivation:

$$C(t) \equiv \langle L' F' \rangle = \langle L' \bar{V}^0 * L' \rangle \bar{F} + \langle L' \bar{V}^0 * B L' U_0 * L' \rangle \bar{F} \\ + \underbrace{\langle L' \bar{V}^0 * L' U_0 * L' F' \rangle}_{11.11}$$

Actually the last term (equivalent to the Dupree Beta term /2/) is of the type

$$\langle L' L' \rangle \langle L' F' \rangle$$

which involves the complete $C(t)$: Eq. 11.11 still remains an integral equation for C , which describe some kind of renormalization or shielding of the RQL-weak coupling approximation given by

$$C^0 = \langle L' U^0 * L' \rangle \bar{F}$$

The purpose of the present work is to elucidate the role of such higher-order implicit terms in the equation which defines the turbulent collisions.

II.B. Simple calculation

This above schematical introduction of the higher order terms can be made slightly more precise by considering the wave-vector dependence in Fourier transform.

For the case of a turbulence which is homogeneous in average, we have a space-independent average function

$$\bar{f} = \bar{f}(v, t)$$

and

$$\bar{L} = -v \cdot \nabla$$

and

$$L'_k = -\frac{q}{m} E'_k(t) \frac{\partial}{\partial v}$$

since we have no average Vlasov field. In this case (II.3a) writes

$$\partial_t f'_k(t) = -i k \cdot v f'_k(t) + L'_k(t) \bar{f}(t) + \int dk' B L'_{k'}(t) f'_{k-k'}(t) \quad 11.12$$

In order to perform the iteration, we write the formal solution for $f'_{k-k'}$. Neglecting as usual the initial value term, the asymptotic solution writes

$$f'_{k-k'}(t) = \int_{-\infty}^t dt' e^{-i(t-t')(k-k') \cdot v} \left[L'_{k-k'}(t') \bar{f}(t') + \int dk'' B L'_{k''}(t') f'_{k-k'-k''}(t') \right] \quad 11.13$$

and (II.12) can be written in the form:

$$\partial_t f'_k(t) = -i k \cdot v f'_k(t) + L'_k(t) \bar{f}(t) + \int dk' B L'_{k'}(t) \int_{-\infty}^t dt' e^{-i(t-t')(k-k') \cdot v} \left[L'_{k-k'}(t') \bar{f}(t') + \int dk'' B L'_{k''}(t') f'_{k-k'-k''}(t') \right] \quad 11.14$$

Let us consider the structure of this general expression. In the last term, the first B to the left prescribes the fluctuating part of what follows. (The integral $\int dk''$ can be replaced by the discrete summation $\sum_{k''}$ with appropriate factor $(2\pi/L)^3$ which are omitted here for clarity). In a homogeneous system this implies $k \neq 0$. In the same way the second B implies

$k \neq k'$: two values of k'' appear to play a special role:

i) $k'' = k$ which gives $L'_k(t') f'_{-k}(t')$

which is analogous to the Dupree Beta term

ii) $k'' = -k'$ which gives $L'_{-k'}(t') f'_k(t')$

i.e. a phase coherent term.

Neglecting higher non-coherent mode-coupling terms gives:

$$\partial_t f'_k(t) = -i k \cdot v f'_k(t) + L'_k(t) \bar{f}(t) + \sum_{k'} \int_0^t dz e^{-i z(k-k') \cdot v} \left[L'_k(t) L'_{k-k'}(t-z) \bar{f}(t-z) + L'_k(t-z) \langle L'_{k'}(t) f'_{-k'}(t-z) \rangle + \langle L'_k(t) L'_{-k'}(t-z) \rangle f'_{k'}(t-z) \right]$$

II.15

Here the k' summation is an approximate form for the complete mode-coupling term $\mathcal{B} L' f'$ in II.3a or 11.12 :

$$\mathcal{B} L' f' \rightarrow \sum_{k'} U_0 * [L'_k L'_{k-k'} \bar{f} + L'_k \langle L'_{k'} f'_{-k'} \rangle + \langle L'_k L'_{-k'} \rangle f'_{k'}] \quad \text{II.16}$$

Let us consider the structure of this equation:

- The term $\langle L'_k f'_{-k'} \rangle$ appears as a modification of the source term in \bar{f} : it has been interpreted by Dupree as a modification of the average function \bar{f} into an "effective" average function :

$$\bar{F} = \bar{f} + \langle L'_k f'_{-k'} \rangle$$

- The term $\langle L'_k L'_{-k'} \rangle$ appears as the weak-coupling approximation of the turbulent collision term (coherent part).

In Ref. /3/ these both terms appear as

$$\mathcal{B} L' f' \rightarrow C_k^F f'_k = C_k^F f'_k + \int_0^t dt' L'_k(t') C_k^\phi(t-t')$$

II.17

i.e. a phase coherent term $C^F f'$ (turbulent collisions) and the Beta term (C^ϕ). This corresponds here to the turbulent collisions

$$C_k^F f'_k = \sum_{k'} \langle L'_k U_0 * L'_{-k'} \rangle f'_{k'} \quad \text{11.18}$$

and the Beta term

$$\int_0^t dt' L'_k(t') C_k^\phi(t-t') = \int_0^t dt' \sum_{k'} L'_k(t') U_0(t,t') \langle L'_{k'}(t) f'_{-k'}(t') \rangle \quad \text{II.19}$$

The role of the term $L'_k L'_{k-k'} \bar{f}$ in Dupree's description is not apparent .

III. EXACT KINETIC RESULTS

The appearance of turbulent collisions and higher order terms can actually be described more rigorously by using general expressions which have been obtained in the framework of our kinetic theory for plasma turbulence.

III.A. General kinetic equation for turbulent plasmas

In the kinetic regime of long times the exact solution of II.3a for the fluctuations can be written /4/

$$\begin{aligned} f'(x, v, t) &= \int_{-\infty}^t dt' \Lambda(t, t') L'(t') \bar{f}(x, v, t') \\ &= \int_0^{\infty} dv \Lambda(t, t-v) L'(t-v) \bar{f}(x, v, t-v) \end{aligned} \quad 111.1$$

This is a basic formula which describes the fluctuations in terms of the average distribution function. It holds in the kinetic regime of times long compared to the correlation time of electric field fluctuations; in such regime the influence of the initial fluctuation has been shown to vanish exactly /4/. Here Λ is the Weinstock propagator defined by the homogeneous part of Eq. (II.3a) :

$$\partial_t \Lambda(t, t_0) = [\bar{L}(t) + BL'(t)] \Lambda(t, t_0) \quad 111.2$$

i.e. /5/ :

$$\Lambda(t, t_0) = X_+ e^{\int_{t_0}^t [\bar{L}(t') + BL'(t')] dt'}$$

111.3

with $\Lambda(t_0, t_0) = 1 = \Lambda(t, t)$

111.4

and where X_+ is the time-ordering operator /5/ which prescribes that in the expansion of the exponential operator in power of its exponent, the various operators have to be time-ordered in order of decreasing times when $t \geq t_0$.

Solution 111.1 can be checked immediately: by using (III.2), Eq. (111.1) gives (II.3a) (see 111.4).

This simple exact solution allows us to write the average equation (II.2) as

$$\partial_t \bar{f} = [\bar{L}(t) + G(t)] \bar{f}(t)$$

III.5

$$G(t) = \int_0^t dt' \langle L'(t) \wedge(t, t') L'(t') \rangle \bar{V}(t', t)$$

equation itself:

$$\partial_t \bar{V}(t, t_0) = [\bar{L}(t) + G(t)] \bar{V}(t, t_0)$$

111.7

i.e.

$$\bar{V}(t, t_0) = X_+ e^{\int_{t_0}^t dt' [\bar{L}(t') + G(t)]}$$

111.8

This operator allows us to write the exact solution as

$$\bar{f}(t) = \bar{V}(t, t') \bar{f}(t')$$

III.9

III.B--Iteration

Let us now perform a kind of iteration of Eq. (III.1). A simple integral relation can be obtained between the Weinstock propagator \wedge (III.2) and the turbulent propagator (III.7):

$$\wedge = \bar{V} + \bar{V} * [BL' - G] \wedge = \bar{V} + \wedge * [BL' - G] \bar{V}$$

III.10a

i.e.

$$\Lambda(t, t-z) = \bar{V}(t, t-z) + \int_0^z d\theta \bar{V}(t, t-\theta) [BL'(t-\theta) - G(t-\theta)] \Lambda(t-\theta, t-z)$$

III.10b

This integral equation can be deduced from (III.2) by treating $(BL' - G)$ as a perturbation:

$$\partial_t \Lambda = [\bar{L} + G + \frac{1}{2} BL'G] \Lambda$$

By substituting (III.10) into (III.1) we immediately obtain the following iterated formula for the fluctuating distribution function:

$$\begin{aligned} F'(t) &= \int_0^\infty dz \bar{V}(t, t-z) L'(t-z) \bar{F}(t-z) \\ &+ \int_0^\infty dz \int_0^z d\theta \bar{V}(t, t-\theta) [BL'(t-\theta) - G(t-\theta)] \Lambda(t-\theta, t-z) L'(t-z) \bar{F}(t-z) \end{aligned}$$

III.11

By substituting this exact solution in (II.2) we find a new expression for the turbulent collision term:

$$C(t) \equiv \langle L'F' \rangle = G(t) \bar{F}(t) = \int_0^\infty dz \langle L'(t) \Lambda(t, t-z) L'(t-z) \rangle \bar{V}(t-z, t) \bar{F}(t)$$

III.12

which is given by

$$\begin{aligned} C(t) &= \int_0^\infty dz \langle L'(t) \bar{V}(t, t-z) L'(t-z) \rangle \bar{F}(t-z) \\ &+ \int_0^\infty dz \int_0^z d\theta \langle L'(t) \bar{V}(t, t-\theta) [BL'(t-\theta) - G(t-\theta)] \Lambda(t-\theta, t-z) L'(t-z) \rangle \bar{F}(t-z) \end{aligned}$$

III.13

Introducing the variable $S = \tau - \theta$ we have

$$\int_0^\infty d\tau \int_0^\tau d\theta = \int_0^\infty d\theta \int_\theta^\infty d\tau = \int_0^\infty d\theta \int_0^\infty dS$$

and finally the result can be written as ($\theta \rightarrow \tau$):

$$C(t) = \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \bar{F}(t) \\ + \int_0^\infty d\tau \int_0^\tau dS \langle L'(t) \bar{V}(t, t-\tau) [BL'(t-\tau) - G(t-\tau)] \Lambda(t-\tau, t-\tau-S) L'(t-\tau-S) \rangle \bar{V}(t-\tau-S, t) \bar{F}(t)$$

111.14

The first term is nothing else than the RQL approximation of the collision term, which has been developed in plasma turbulence in analogy with the DIA approximation in fluid turbulence/6/. This approximation is obtained by replacing in the exact expression (111.6) the fluctuating trajectories described by the Weinstock propagator Λ by the average turbulent trajectories described by the turbulent propagator \bar{V} . The second term in (III.14) takes into account the deviation from the RQL description, by means of $(BL' - G)$.

III.C. Higher order terms beyond-RQL

In order to make a bridge with the more intuitive calculation of Section II, it is interesting to put this correction in another form. From (III.1) we have indeed:

$$F'(t-\tau) = \int_0^\infty dS \Lambda(t-\tau, t-\tau-S) L'(t-\tau-S) \bar{V}(t-\tau-S, t-\tau) \bar{F}(t-\tau)$$

III.15

Since

$$\bar{F}(t-\tau-S) = \bar{V}(t-\tau-S, t-\tau) \bar{F}(t-\tau) = \bar{V}(t-\tau-S, t) \bar{F}(t)$$

111.16

we can write (III.14) as :

$$\begin{aligned}
C(t) = G(t) \bar{F}(t) = & \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \bar{F}(t) \\
& + \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) B L'(t-\tau) \int_0^\infty ds \Lambda(t-\tau, t-\tau-s) L'(t-\tau-s) \rangle \bar{V}(t-\tau-s, t) \bar{F}(t) \\
& - \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) G(t-\tau) f'(t-\tau) \rangle
\end{aligned}$$

111.17

When compared with 11.11 the present exact result (111.17) can be analyzed as follows. The first term is the RQL approximation of the collision term, while the first term in 11.11 only represents the quasi-linear result ($\Lambda \rightarrow U_0$). The second term in (111.17) is a generalization of the second term of (II.11) i.e., a term involving at least three fluctuating fields; here

Λ remains a fluctuating quantity which may introduce correlations between four fields, a.s.o. The third term in (III.17) is the general form of the Beta term obtained in (II.11). The present general formulation avoids using approximate treatments of averages like that denoted by the arrow in Eq. (II.11) and avoids elimination of higher order mode-coupling effects.

Moreover it allows to exhibit the implicit feature of this higher order Beta term: the deviation of G from the RQL approximation of G involves a Beta term which depends on G itself; this was already apparent in Eq. (II.11) where $\langle L'f' \rangle$ was given in terms of $\langle L'f' \rangle$.

In summary, we have shown that the turbulent collision term involves a first lowest order approximation (QL in 11.11, RQL in III.17) plus higher order terms which have been calculated. This latter correction involves an important, so-called Beta term, which actually depends on the complete collision term ("implicit" terms).

III.D. Approximations

Two levels of approximation can be used to describe (111.14).

- 1°. The RQL approximation for G is solution of the non-linear equation (replace Λ by \bar{V} in III.12):

$$G_{RQL} = \int_0^\infty d\tau \langle L'(t) \bar{V}_{RQL}(t, t-\tau) L'(t-\tau) \rangle \bar{V}_{RQL}(t-\tau, t) \quad \text{III.18}$$

with

$$\bar{V}_{RQL}(t, t-\tau) = X_+ e^{\int_{t-\tau}^t dt' [\bar{L}(t') + G_{RQL}(t')]} \quad \text{III.19}$$

- 2°. Higher order corrections can be obtained by replacing Λ by \bar{V} in the iterated formula obtained in (III.14). In this higher approximation we have

$$C(t) = \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \bar{F}(t) + \int_0^\infty d\tau \int_0^\infty ds \langle L'(t) \bar{V}(t, t-\tau) [\bar{L}(t-\tau) - G(t-\tau)] \bar{V}(t-\tau, t-\tau-s) L'(t-\tau-s) \rangle \bar{V}(t-\tau-s, t) \bar{F}(t) \quad \text{III.20}$$

This equation is the main tractable result of the present work, It represents a highly non-linear equation for the collision term of the average equation

$$\partial_t \bar{F} = \bar{L} \bar{F} + C(t)$$

described in an approximation which goes beyond the RQL description. The calculation of an approximate solution for this new closed equation is left for future work.

IV. NON-ASYMPTOTIC TREATMENT FOR TURBULENT COLLISIONS

An integral equation similar to (III.17) for the collision term of Eq. (II.2) has been obtained from the Vlasov equation directly /6/. We will show here that the physical content of the different terms appear to be quite different.

IV. A. Integral equation

The fluctuating equation (II.3) can be written in the form

$$\partial_t F' = [L + L'] F' + L' \bar{F} - \langle L' F' \rangle$$

IV.1

This form naturally introduces the exact propagator U defined by

$$\partial_t U(t, t_0) = [L(t) + L'(t)] U(t, t_0)$$

IV.2

i.e.

$$U(t, t_0) = X_+ e^{\int_{t_0}^t dt' [L(t') + L'(t')]}$$

IV.3

Contrary to the Weinstock propagator, the exact propagator

U does not prescribe all intermediate states to be fluctuating ones: U allows "transitions" from an average "state" to a fluctuating one, and vice-versa. The Weinstock propagator Λ (III.3) in turns describes the irreducible propagation of fluctuating states /4/.

Solution of (IV.1) can be written (neglecting the initial fluctuation):

$$F' = U * L' \bar{F} - U * \langle L' F' \rangle$$

IV.4

i.e.

$$F'(t) = \int_{t_0}^t dt' U(t, t') [L'(t') \bar{F}(t') - \langle L'(t') F'(t') \rangle]$$

IV.5

$$\begin{aligned} \langle L'F' \rangle = C(t) = & \int_0^{t-t_0} d\tau \langle L'(t) U(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \\ & - \int_0^{t-t_0} d\tau \langle L'(t) U(t, t-\tau) \rangle C(t-\tau) \end{aligned} \quad \text{IV. 6}$$

IV.B. Three contributions to the collision term.

Let us transform Eq. (IV.6) in order to introduce explicitly a first term which in the asymptotic limit ($t_0 \rightarrow -\infty$) gives the RQL approximation. For this, we simply decompose

$$U = \bar{U} + U' \quad \text{IV. 7}$$

where the average \bar{U} of the exact propagator actually tends to the average turbulent propagator \bar{V} (III.8) in the limit: of time intervals long compared to the correlation time of electric field fluctuations. Then (IV.6) becomes:

$$C(t) = C_I(t) + C_{II}(t) + C_{III}(t) \quad \text{IV. 8}$$

where the three contributions are given by

$$C_I(t) = \int_0^{t-t_0} d\tau \langle L'(t) \bar{U}(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \xrightarrow{t_0 \rightarrow -\infty} C_{RQL} \quad \text{IV. 9}$$

$$C_{II}(t) = \int_0^{t-t_0} d\tau \langle L'(t) U'(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \quad \text{IV. 10}$$

$$C_{III}(t) = - \int_0^{t-t_0} d\tau \langle L'(t) U'(t, t-\tau) \rangle C(t-\tau) \quad \text{IV. 11}$$

IV.C. Identities between various propagators.

In order to build a bridge between the present treatment and the asymptotic results of Section III, we have to calculate U' in terms of Λ and \bar{V} . For this, we have

to derive from the definitions of the propagators some simple exact relations in the form of integral equations.

Let us first derive the (integral) relation between U and U_0 :

$$\underline{U = U_0 + U_0 * L' U}$$

IV.12

which is readily obtained from (IV.2) and (II.4). We have to note that in such relations, the two propagators in the second term can be equally inverted since both formulas give the same iterated explicit solution :

$$U = \sum_{n=0}^{\infty} (U_0 * L')^n U_0$$

IV.13

Let us now express U_0 in terms of \bar{V} and U_0 . From (III.7) and (II.4) we have:

$$\underline{\bar{V} = U_0 + \bar{V} * G U_0}$$

IV.14

Combining this with (IV.12) gives

$$U = U_0 + \bar{V} * L' U - \bar{V} * G U_0 * L' U$$

IV.15

Let us also express U in terms of Λ . Eq.(III.2) can be written:

$$\partial_t \Lambda = [\bar{L} + L' - A L'] \Lambda$$

which gives

$$\underline{\Lambda = U - U * A L' \Lambda}$$

IV.16

Then IV.15 gives

$$U = U_0 + \bar{V} * L' \Lambda + \bar{V} * L' U * A L' \Lambda - \bar{V} * G U_0 * L' U$$

IV.17

Here U_0 in the last term can be eliminated by 'using (IV.12 and 16) :

$$U_0 * L' U = U - U_0 = \Lambda + U * A L' \Lambda - U_0$$

IV.18

and we finally obtain

$$U = U_0 + \bar{V} * L' \Lambda + \bar{V} * L' U * A L' \Lambda - \bar{V} * G [U - U_0] * L' U$$

IV.19a

$$U = U_0 + \bar{V} * L' \Lambda + \bar{V} * L' U * A L' \Lambda - \bar{V} * G [\Lambda - U_0 + U * A L' \Lambda]$$

IV.19b

The fluctuating part of U can thus be written as

$$U' = B \left\{ \bar{V} * L' \Lambda + \bar{V} * L' U * A L' \Lambda - \bar{V} * G \Lambda - \bar{V} * G U * A L' \Lambda \right\}$$

i.e.

$$\boxed{U' = \bar{V} * [B L' - G] \Lambda + \bar{V} * [B L' - G] U * A L' \Lambda}$$

Iv. 20a

or, explicitly:

$$U'(t, t-z) = \int_0^z d\theta \bar{V}(t, t-\theta) [B L'(t-\theta) - G(t-\theta)] \Lambda(t-\theta, t-z) \\ + \int_0^z d\theta \bar{V}(t, t-\theta) [B L'(t-\theta) - G(t-\theta)] \int_0^z ds U(t-\theta, t-s) A L'(t-s) \Lambda(t-s, t-z)$$

IV.20b

IV.D. Asymptotic cancellation of C_{III}

From the exact relation (IV.20) we can write the second contribution to the collision term (IV.10) in the form:

$$C_{II}(t) = \int_0^{t-t_0} d\tau \langle L'(t) [\bar{U} + U'] L'(t-z) \rangle \bar{F}(t-z) = C_{II_1} + C_{II_2}$$

IV.21

with

$$C_{II_1}(t) = \int_0^{t-t_0} d\tau \langle L'(t) \int_0^z d\theta \bar{V}(t, t-\theta) [B L'(t-\theta) - G(t-\theta)] \Lambda(t-\theta, t-z) L'(t-z) \rangle \bar{F}(t-z)$$

IV.22

$$C_{II_2}(t) = \int_0^{t-t_0} d\tau \langle L'(t) \int_0^\tau d\theta \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] \int_0^\tau ds U(t-\theta, t-s) \rangle \quad \text{IV.23}$$

The total collision term (IV.8) thus writes:

$$C(t) = C_I(t) + C_{II_1}(t) + C_{II_2}(t) + C_{III}(t) \quad \text{IV.24}$$

Actually one can prove that, in the asymptotic limit, two last terms in (IV.24) exactly cancel (see Appendix) :

$$C_{II_2}^{As.}(t) + C_{III}^{As.}(t) = 0 \quad \text{IV.25}$$

This implies that we have asymptotically:

$$C(t) \longrightarrow \lim_{t_0 \rightarrow -\infty} [C_I(t) + C_{II_1}(t)] \quad \text{IV.26}$$

Now this result can be proved to be identical to Eq. (III.14) obtained in the asymptotic description. This identity can be proved as follows. From (IV.9) we have indeed

$$\lim_{t_0 \rightarrow -\infty} C_I(t) = \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \quad \text{IV.27}$$

which is indeed the first term of (III.14). On the other hand, we have from (IV.22):

$$\begin{aligned} \lim_{t_0 \rightarrow -\infty} C_{II_1}(t) &= \int_0^\infty d\tau \int_0^\tau d\theta \langle L'(t) \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] \wedge(t-\theta, t-\theta) L'(t-\theta) \rangle \bar{F}(t-\theta) \\ &= \int_0^\infty d\theta \int_0^\infty ds \langle L'(t) \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] \wedge(t-\theta, t-\theta-s) L'(t-\theta-s) \rangle \bar{F}(t-\theta-s) \end{aligned} \quad \text{IV.28}$$

which is identical to the second term of (III.14). This achieves the demonstration of the equivalence of (IV.26) and (III.14).

We have thus seen in the general result (IV.24) that in the asymptotic limit of a distant initial condition, last two terms exactly cancel. This means that in (IV.6) the last implicit term is cancelled asymptotically by a part of

the contribution of U' in the first explicit term, and we remain with the asymptotic result given by the kinetic formulation (III.14). This remains a highly non-linear equation for the collision term, even in the approximation (III.20) where Λ has been replaced by \bar{V} .

V. ANALYSIS OF THE LAGRANGIAN AUTOCORRELATION OF FIELDS.

V.A. Lagrangian autocorrelation for turbulent plasmas and Brownian motion.

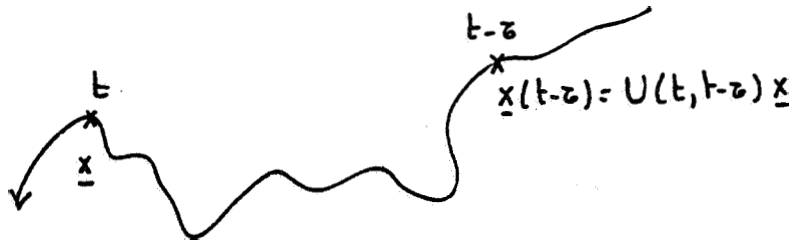
Let us consider the exact relation (IV.6) which has been derived for the turbulent collision term from the Vlasov or Klimontovich equation:

$$C(t) = \langle L'F \rangle = C_L(t) - \int_0^{t-t_0} \langle L'(t) U(t, t-\tau) \rangle C(t-\tau) \quad \text{V. 1}$$

where the first term C_L is defined in terms of the Lagrangian correlation between two electric field fluctuations along the exact motion of the particle, which is described by the propagator U :

$$C_L(t) = C_I(t) + C_{II}(t) = \int_0^{t-t_0} \langle L'(t) U(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \\ = \frac{q^2}{m^2} \frac{\partial}{\partial \underline{v}} \cdot \int_0^{t-t_0} \langle \underline{E}'(\underline{x}, t) U(t, t-\tau) \underline{E}'(\underline{x}, t-\tau) \rangle \cdot \frac{\partial}{\partial \underline{v}} \bar{F}(\underline{x}, \underline{v}, t-\tau) \quad \text{v.2}$$

This autocorrelation of the electric field fluctuation can be visualized as the correlation between two points of the space-time (\underline{x}, t) which are defined by the exact position of a particle which is at point \underline{x} at time t and at a point $\underline{x}(t-\tau) = U(t, t-\tau)\underline{x}$ at time $t-\tau$ (see Fig.1).



The time integral of this autocorrelation is fully similar to the well-known autocorrelation of fluctuating forces , which

appears in the classical description of Brownian motion. The friction coefficient of the Langevin equation

$$\frac{d\mathbf{p}}{dt} = -\frac{\zeta}{m} \mathbf{p} + \mathbf{X}(t) = \mathbf{F}(t)$$

v. 3

is related to the diffusion coefficient by the Einstein relation

$$D = \frac{kT}{\zeta}$$

v. 4

and is given by the time integral of the autocorrelation of the total fluctuating force $\mathbf{F}(t)$ (see p. 262 in /7/):

$$\zeta = \frac{1}{3kT} \int_0^T dt \langle \mathbf{F}(t) \cdot \mathbf{F}(t+\tau) \rangle$$

v. 5

Here the upper integration limit? is larger than the correlation time τ_c of the fluctuations, but smaller than the relaxation time m/ζ :

$$\tau_c < T < \tau_r = \frac{m}{\zeta}$$

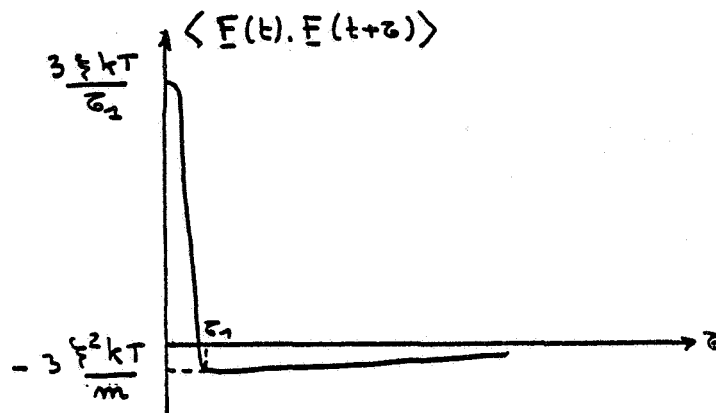
v. 6

The point we want to stress here is the following: the exact autocorrelation has a strong peak near the origine ($\tau \lesssim \tau_c$) but also a long negative tail:

$$\langle \mathbf{F}(t) \cdot \mathbf{F}(t+\tau) \rangle = 6\zeta kT \delta(\tau) - \frac{3\zeta^2 kT}{m} e^{-\frac{\zeta\tau}{m}} + \left(\frac{\zeta}{m}\right)^2 e^{-\frac{\zeta}{m}(2t+\tau)} (p_0^2 - 3mkT)$$

v. 7

(see Fig. 2 /8/)



which makes that the total integral (from zero to infinity) of the autocorrelation is actually zero:

$$\int_0^{\infty} d\tau \langle E(t) \cdot E(t+\tau) \rangle = 0$$

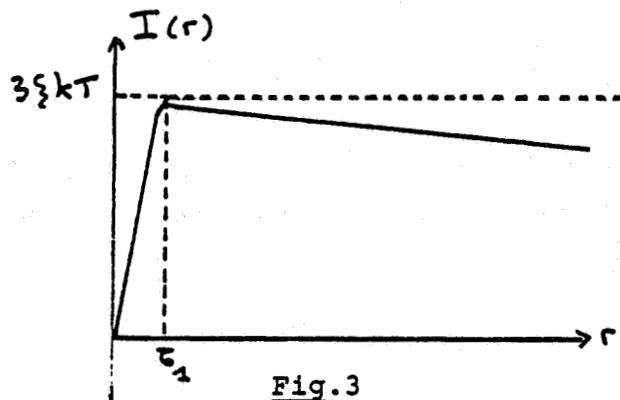
V. 8

The finite integral however

$$I(r) = \int_0^r d\tau \langle E(t) \cdot E(t+\tau) \rangle$$

V. 9

is a function of r which reaches a quasi-plateau value at (see p. 261 in /7/)



V.B. -- General result

The point we want to show here is actually very similar. Let us consider Eq. (V.1) and calculate the second term. From the exact equation for U we have:

$$\partial_t U = L U$$

V. 10

thus

$$\partial_t \bar{U} = \bar{L} \bar{U} + \langle L' U' \rangle$$

V. 11

This can also be written

$$\langle L'(t) U'(t, t') \rangle = \left[\frac{\partial}{\partial t} - \bar{L}(t) \right] \bar{U}(t, t')$$

V. 12

When substituted into (V.1) :

$$C(t) - C_L(t) = - \int_{t_0}^t dt' \langle L'(t) U'(t, t') \rangle C(t') = C_{III}(t)$$

V. 13

this gives an equation where fluctuating quantities have been eliminated:

$$C_{III}(t) = - \int_{t_0}^t dt' \left[\frac{\partial}{\partial t} - \bar{L}(t) \right] \bar{U}(t, t') C(t') \quad V.14$$

By using partial integration

$$\int_{t_0}^t dt' \frac{\partial}{\partial t} H(t, t') = \frac{\partial}{\partial t} \int_{t_0}^t dt' H(t, t') - H(t, t) \quad V.15$$

we obtain

$$C_{III}(t) = C(t) - \left[\frac{\partial}{\partial t} - \bar{L}(t) \right] \int_{t_0}^t dt' \bar{U}(t, t') C(t') \quad V.16$$

By this way we have proved that

$$C_L(t) = \left[\frac{\partial}{\partial t} - \bar{L}(t) \right] \int_{t_0}^t dt' \bar{U}(t, t') C(t') \quad V.17$$

This is an integral equation for C in terms of C_L which is given by the time integral of the Lagrangian autocorrelation of the fields. Here the integral operator has to be inverted in order to determine C in terms of C_L . An important conclusion can nevertheless be reached for a particular homogeneous case.

V.C. Asymptotic description for homogeneous and stationnary turbulence

In the asymptotic case ($t_0 \rightarrow -\infty$) this equation becomes:

$$C_L^{As}(t) = \int_0^\infty d\tau \langle L'(t) U(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) = \left[\frac{\partial}{\partial t} - \bar{L}(t) \right] \int_0^\infty d\tau \bar{U}_{As}(t, t-\tau) C(t-\tau) \quad V.18$$

and

$$\bar{U}_{As}(t, t-\tau) = \bar{V}(t, t-\tau) = X_+ e^{\int_{t-\tau}^t dt' [\bar{L}(t') + G(t')]} \quad V.19$$

In the stationary case $\bar{V}(t, t-\tau) = \bar{V}(t)$ is t -independent, and $C(t, \tau)$ becomes time-independent. We remain with

$$C_L^{As}(t) = - \bar{L}(t) \int_0^\infty d\tau \bar{V}(\tau) C \quad \text{v.20}$$

For homogeneous systems, C and \bar{V} are x -independent and $\bar{L} = -\gamma \cdot \nabla$ (no average Vlasov field). We thus reach the result that for an homogeneous and stationary turbulence the infinite time integral of the Lagrangian autocorrelation of fluctuating fields vanishes in the asymptotic description:

$$C_L^{As}(t) = C_I^{As} + C_{II}^{As} = \frac{q^2 \gamma}{m^2 \omega_V} \cdot \int_0^\infty d\tau \langle \underline{E}'(t) U(t, t-\tau) \underline{E}'(t-\tau) \rangle \cdot \frac{\gamma}{\omega_V} \bar{F}(t-\tau) \rightarrow 0$$

v.21

although the total turbulent collision term does not vanish. This result is very similar to Eq.(V.8) for Brownian motion.

V.D. Consequence for turbulent collisions

This result implies that

$$C_{II}^{As} = C^{As} - C_L^{As} = C^{As} \quad \text{v.22}$$

It is interesting to combine this result with (IV.25) :

$$C_{II_2}^{As} = - C_{III}^{As} = - C^{As} \quad \text{v.23}$$

When reported in Eq.(IV.24) this gives

$$C^{As}(t) = \underbrace{C_I^{As}(t) + C_{II_2}^{As}(t)}_{C(t)} + \underbrace{C_{II_2}^{As}(t)}_{-C(t)} + \underbrace{C_{III}^{As}(t)}_{+C(t)}$$

v.24

This means that in the asymptotic Analysis of the four terms appearing in (IV.8) for the collision term :

1°) last C_{III} Contribution (IV.11) goes to the complete result:

$$C_{III}^{As}(t) = - \int_0^\infty d\tau \langle L'(t) U'(t, t-\tau) \rangle C(t-\tau) = C(t) \quad \text{V.25}$$

2°) The contribution C_I (IV.10) goes to $-C_I$:

$$\begin{aligned} C_{II}^{As}(t) &= \int_0^\infty d\tau \langle L'(t) U'(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) \\ &= - \int_0^\infty d\tau \langle L'(t) \bar{U}(t, t-\tau) L'(t-\tau) \rangle \bar{F}(t-\tau) = -C_I^{As}(t) \end{aligned} \quad \text{V.26}$$

3°) The total collision term is given by the result of the kinetic method developed in Section III :

$$C^{As}(t) = C_I^{As} + C_{II_1}^{As} = C_{III}^{As} = -C_{II_2}^{As}$$

V. 27

$$\begin{aligned} \text{i.e. :} \quad C(t) &= \int_0^\infty d\tau \langle L'(t) \bar{V}(t, t-\tau) L'(t-\tau) \rangle \bar{V}(t-\tau, t) \bar{F}(t) \\ &+ \int_0^\infty d\tau \int_0^\infty ds \langle L'(t) \bar{V}(t, t-\tau) [B L'(t-\tau) - G(t-\tau)] \Lambda(t-\tau, t-\tau-s) L'(t-\tau-s) \rangle \bar{F}(t-\tau-s) \end{aligned} \quad \text{V.28}$$

These equation (V.25,26,28) are three independent relations. The importance of the last result is that it allows us to describe the turbulent collision term by a modification of the RQL approximation. The contribution of G in (V.28) represents the equivalent of the Dupree Beta terms, here in a general form. Their importance has been proved for energy conservation. We will come back later to this important point.

Let us prove the cancellation of C_{III} and C_{II_2} in the asymptotic case. We calculate (IV.11)

$$C_{III} = - \int_0^{t-t_0} d\tau \langle L'(t) U'(t, t-\tau) \rangle C(t-\tau) \quad A.1$$

and use the exact asymptotic expression for C (III.12):

$$C_{III}^{As}(t) = \lim_{T \rightarrow \infty} - \int_0^T d\tau \langle L'(t) U'(t, t-\tau) \rangle \int_{\bar{\tau}}^T d\tau' \langle L'(t-\tau) \Lambda(t-\tau, t-\tau') L'(t-\tau') \rangle \bar{F}(t-\tau') \quad A.2$$

Let us modify the variables:

$$\int_0^T d\tau \int_{\bar{\tau}}^T d\tau' = \int_0^T d\tau' \int_0^{\tau'} d\tau = \int_0^T d\tau' \int_0^{\tau'} ds \quad A.3$$

and eliminate τ by the new variables $s = \tau' - \tau$. We obtain:

$$C_{III}^{As} = \lim_{T \rightarrow \infty} - \int_0^T d\tau' \int_0^{\tau'} ds \langle L'(t) U'(t, t-\tau'+s) \rangle \langle L'(t-\tau'+s) \Lambda(t-\tau'+s, t-\tau') L'(t-\tau') \rangle \bar{F}(t-\tau') \quad A.4$$

Let us now use (V.20b) for U' :

$$U'(t, t-\tau'+s) = \int_0^{\tau'-s} d\theta \bar{V}(t, t-\theta) [B L'(t-\theta) - G(t-\theta)] \left\{ \Lambda(t-\theta, t-\tau'+s) + \int_{\theta}^{\tau'-s} d\tau U(t-\theta, t-\tau) A L'(t-\tau) \Lambda(t-\tau, t-\tau'+s) \right\} \quad A.5$$

The bracket can actually be related to the exact propagator

U by means of (IV.16):

$$U(t-\theta, t-\tau'+s) = \Lambda(t-\theta, t-\tau'+s) + \int_{t-\tau'+s}^{t-\theta} dt' U(t-\theta, t') A L'(t') \Lambda(t', t-\tau'+s) \\ = \Lambda(t-\theta, t-\tau'+s) + \int_{\theta}^{\tau'-s} d\tau U(t-\theta, t-\tau) A L'(t-\tau) \Lambda(t-\tau, t-\tau'+s) \quad A.6$$

(where $p = t, t'$) . Substituting (A.6) into (A.5) gives

$$U'(t, t-z'+s) = \int_0^{z'-s} d\theta \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] U(t-\theta, t-z'+s) \quad A.7$$

When substituted into (A.4) this result allows us to express

$$C_{II}^{As} \text{ as } : (\text{change } z' \text{ into } z) \\ C_{II}^{As}(t) = \lim_{T \rightarrow \infty} - \int_0^T dz \int_0^z ds \int_0^{z-s} d\theta \langle L'(t) \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] U(t-\theta, t-z+s) \rangle \\ \langle L'(t-z+s) \wedge (t-z+s, t-z) L'(t-z) \rangle \bar{F}(t-z) \quad A.8.$$

On the other hand, C_{II_2} given by (IV.23) can also be transformed by using

$$\int_0^z d\theta \int_0^z ds \stackrel{s=z-z'}{=} \int_0^z d\theta \int_0^{z-\theta} dz' = \int_0^z dz' \int_0^{z-z'} d\theta \quad A.9$$

and we obtain ($z' \rightarrow s$) :

$$C_{II_2}(t) = \int_0^T dz \langle L'(t) \int_0^z ds \int_0^{z-s} d\theta \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] U(t-\theta, t-z+s) \rangle \\ \langle L'(t-z+s) \wedge (t-z+s, t-z) L'(t-z) \rangle \bar{F}(t-z) \quad A.10$$

By comparing with (A.8) we have thus demonstrated the result (IV.25) :

$$C_{II}^{As}(t) = - C_{III}^{As}(t) \\ = \int_0^\infty dz \int_0^z ds \int_0^{z-s} d\theta \langle L'(t) \bar{V}(t, t-\theta) [BL'(t-\theta) \cdot G(t-\theta)] U(t-\theta, t-z+s) \rangle \\ \langle L'(t-z+s) \wedge (t-z+s, t-z) L'(t-z) \rangle \bar{F}(t-z) \quad A.11$$

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SECTION 5

SPECTRAL STRUCTURE OF TURBULENCE IN THE STABLE ATMOSPHERIC BOUNDARY LAYER

C.M. Tchen

GRADUATE CENTER AND CITY COLLEGE
OF THE CITY UNIVERSITY OF NEW YORK, N.Y. 10031, U.S.A.

and

S.E. Larsen, H. Pecceili, T. Mikkelsen
RISØ NATIONAL LABORATORY, DK-4000 ROSKILDE, DENMARK

ABSTRACT

The hydrodynamical equations of turbulence are transformed into a master equation for the velocity distribution function. A group-scaling is introduced for the closure. The spectral balance for the velocity fluctuations of individual components shows that the scaled pressure-strain correlation and the cascade-transfer are two transport functions that play the most important roles. We derive this correlation and find a power spectrum k^{-3} for the horizontal components, while the spectrum for the vertical component drops rapidly by going to the large scales.

In a strongly stable boundary layer, the spectral distributions of the large scale horizontal components take the power law k^{-3} , while the vertical component decays rapidly toward the small k values. This spectral hump in the horizontal components are investigated here. We use the group-kinetic method of closure developed by Tchen¹. Instead of the customary Fourier decomposition of a random function, we perform a group-scaling as a coarse-graining procedure. We replace the hydrodynamical equations of turbulence by a master equation for the velocity distribution, and exploit its advantage of being homogeneous and having less nonlinearity, since the turbulent velocity is now an independent variable.

11. BASIC EQUATIONS OF ATMOSPHERIC TURBULENCE

A. Microdynamical State

The microdynamical state of atmospheric turbulence is governed by the equation of continuity and the hydrodynamical equations of Navier-Stokes, as follows:

$$\nabla \cdot \hat{\underline{u}} = 0 \quad (1a)$$

$$\left(\partial_t + \hat{\underline{u}} \cdot \nabla - \nu \nabla^2 \right) \hat{\underline{u}} = \hat{\underline{G}}, \quad \partial_t \equiv \partial/\partial t, \quad \nabla \equiv \partial/\partial \underline{x} \quad (1b)$$

The total function, as denoted by $(\hat{\quad})$, consists of an average quantity, as denoted by

$$(\overline{\quad}) = \langle \quad \rangle,$$

and a fluctuation, or a deviation from the average, as denoted by $(\tilde{\quad})$. The evolution of the velocity $\hat{\underline{u}}$ is controlled by the kinematic viscosity³ and

a field

$$\hat{\underline{\underline{E}}} = \hat{\underline{\underline{E}}} + \hat{\underline{\underline{E}}} \quad (2)$$

which is the sum of the adverse pressure gradient

$$\hat{\underline{\underline{E}}} = - \frac{1}{\rho_0} \nabla \hat{p} \quad (3a)$$

divided by a constant reference density ρ_0 , and the buoyancy

$$\hat{\underline{\underline{E}}} = - \frac{\hat{\rho} - \rho_0}{\rho_0} g \hat{\underline{\underline{j}}} - g \hat{\underline{\underline{j}}} , \quad \hat{\underline{\underline{j}}} = (0, 0, 1) . \quad (3b)$$

The buoyancy is produced by the variable density $\hat{\rho}$ under the gravitational acceleration $\underline{\underline{g}}$. If the temperature \hat{T} and the humidity \hat{Q} have their mean values T , Q , constant reference values T_0 , Q_0 , and fluctuations

θ , q , we can write, under the Boussinesq hypothesis,

$$\frac{\hat{\rho} - \rho_0}{\rho_0} = - \left(\frac{\hat{T} - T_0}{T_0} - \frac{\hat{Q} - Q_0}{Q_0} \right) \quad (4)$$

The temperature and the humidity satisfy the following equations of evolution:

$$\left(\partial_t + \hat{\underline{\underline{u}}} \cdot \nabla - \kappa \nabla^2 \right) \hat{T} = 0 \quad (5a)$$

$$\left(\partial_t + \hat{\underline{\underline{u}}} \cdot \nabla - \kappa \nabla^2 \right) \hat{Q} = 0 , \quad (5b)$$

with the molecular diffusivity κ .

The system of equations (1a), (1b), (5a) and (5b) describes the microdynamical state of stratified turbulence.

B. Group-Scaling of Fluctuations

For the study of the spectral structure, we will be interested in the fluctuations $\tilde{\underline{\underline{u}}}$, $\tilde{\theta}$, \tilde{q} , whose governing equations can be obtained by applying the fluctuation operator $\tilde{A} \equiv 1 - \bar{A}$ to (1) and (5), where $\bar{A} \equiv \langle \rangle$ is the averaging operator. These equations have terms proportional to the mean gradients

$$\nabla u, \quad \nabla T, \quad \nabla \varphi \quad (6)$$

treated as constant parameters. For the sake of simplicity, we combine the temperature and the humidity into a single variable \hat{w} , called drift, such that

$$\nabla \gamma \bar{w} = \nabla g \left(\frac{\bar{T}}{\bar{T}_0} - \frac{\bar{Q}}{\bar{Q}_0} \right) \quad (7a)$$

$$\gamma \tilde{w} = g \left(\frac{\theta}{T_0} - \frac{q}{T_0} \right) . \quad (7b)$$

The constant coefficient γ has the dimension of a frequency, and will always accompany the drift to form the buoyancy force.

The differential equations which govern the fluctuating functions \tilde{u} and \tilde{w} can be obtained by applying the operator \tilde{A} to (1) and (5). **These** equations do not contain the mode-couplings, unless a Fourier transformation is made. However, the Fourier form, like the original equations for the fluctuations, contain too many minute details. A coarse-graining is necessary for a statistical study. For this reason, we introduce a group-scaling, by means of the scaling operators A^0 and A , which compose \tilde{A} , in the form:

$$\tilde{A} = A^0 + A', \quad (8a)$$

and the sequence continues by writing

$$A' = A^{(1)} + A'' . \quad (8b)$$

Thus we have the macro-velocity fluctuation

$$A^0 \hat{u} = u^0, \quad (9a)$$

the micro-velocity fluctuation

$$A' \hat{u} = u', \quad (9b)$$

the submacro-velocity fluctuation

$$A^{(1)} \hat{u} = u^{(1)}, \quad (9c)$$

and the submicro-velocity fluctuation

$$A' \hat{u} = u' . \quad (9d)$$

The groups \underline{u}^0 , \underline{u}' , \underline{u} have their correlation times

$$\tau_c^0 > \tau_c' > \tau_c'' \quad (10)$$

ranged in the decreasing order of magnitude, indicating their decreasing coherence.

By the aid of A^0 , we transform (1a), (1b) and (5) into:

$$(\partial_t + A^0 \underline{\hat{u}} \cdot \nabla - \gamma \nabla^2) \underline{u}_i^0 = - \underline{u}_j^0 \nabla_j \bar{u}_i + \mathcal{E}_i^0 - \nabla_j A^0 \underline{u}_j' \underline{u}_i' \quad (11a)$$

$$(\partial_t + A^0 \underline{\hat{u}} \cdot \nabla - \kappa \nabla^2) \underline{w}_i^0 = - \underline{u}_j^0 \nabla_j \bar{w}_i - \nabla_j A^0 \underline{u}_j' \underline{w}_i' \quad (11b)$$

$$\nabla \cdot \underline{u}^0 = 0, \quad \nabla \cdot \underline{u}' = 0. \quad (11c)$$

It is not difficult to derive the following transport: equations of energy in the group form:

$$\begin{aligned} \frac{1}{2} \partial_t \langle \underline{u}_\alpha^0{}^2 \rangle &= P_{\alpha\alpha}^0 + \langle \underline{u}_\alpha^0 \mathcal{E}_\alpha^0 \rangle - T_{\alpha\alpha}^0 - \mathcal{E}_{\alpha\alpha}^0 \\ &= P_{\alpha\alpha}^0 + \mathcal{E}_{\alpha\alpha}^0 + \varphi_{\alpha\alpha}^0 - T_{\alpha\alpha}^0 - \mathcal{E}_{\alpha\alpha}^0 \end{aligned} \quad (12)$$

$$\frac{1}{2} \partial_t \langle \underline{w}_\alpha^0{}^2 \rangle = P_w^0 - T_w^0 - \mathcal{E}_w^0, \quad (13)$$

with the transport functions:

$$P_{\alpha\alpha}^0 = - \langle \underline{u}_\alpha^0 \underline{u}_j^0 \rangle \nabla_j \bar{u}_\alpha, \quad P_w^0 = - \langle \underline{w}_\alpha^0 \underline{u}_j^0 \rangle \nabla_j \bar{w}_\alpha \quad (14a)$$

$$\mathcal{E}_{\alpha\alpha}^0 = \langle \underline{u}_\alpha^0 \mathcal{E}_\alpha^0 \rangle = \gamma \langle \underline{u}_\alpha^0 \underline{w}_\alpha^0 \rangle \quad (14b)$$

$$\varphi_{\alpha\alpha}^{\circ} \equiv \frac{1}{\rho} \langle \bar{p}^{\circ} \nabla_{\alpha} u_{\alpha}^{\circ} \rangle \quad (14c)$$

$$T_{\alpha\alpha}^{\circ} \equiv \langle u_{\alpha}^{\circ} \nabla_j A^{\circ} u_j^{\circ} u_{\alpha}^{\circ} \rangle, \quad T_w^{\circ} \equiv \langle w_{\alpha}^{\circ} \nabla_j A^{\circ} u_j^{\circ} w_{\alpha}^{\circ} \rangle \quad (14d)$$

$$\varepsilon_{\alpha\alpha}^{\circ} \equiv \nu \langle (\nabla_j u_{\alpha}^{\circ})^2 \rangle, \quad \varepsilon_w^{\circ} \equiv \mu \langle (\nabla_j w_{\alpha}^{\circ})^2 \rangle. \quad (14e)$$

Thus we have the production functions $P_{\alpha\alpha}^{\circ}$ and P_w° by wind shear $\nabla_j u_{\alpha}$ and by mean drift gradient $\nabla_j \bar{w}_{\alpha}$, the buoyancy transport $b_{\alpha\alpha}^{\circ}$, the re-distribution $\varphi_{\alpha\alpha}^{\circ}$ among the components $\alpha = 1, 2, 3$, the cascade transfers $T_{\alpha\alpha}^{\circ}$ and T_w° , and the dissipations $\varepsilon_{\alpha\alpha}^{\circ}$ and ε_w° . The summation rule which applies to the repeating Roman indices does not apply to the Greek indices. The transport equations have been approximated, by treating the mean gradients

$$\nabla \bar{u}, \quad \nabla \bar{w} \quad (15)$$

as constant parameters and by omitting the terms

$$\nabla \langle \dots \rangle$$

as arising from the inhomogeneity of turbulence.

Unlike the non-scaled form, the energy equations of the macro-group contain the mode-couplings by the presence of the transfer functions. We also see the merit in these equations of being able to determine the spectral densities $F_{\alpha\alpha}(k)$ and $F_w(k)$ from the scaled energies

$$\frac{1}{2} \langle u_{\alpha}^{\circ 2} \rangle = \int_0^k dk' F_{\alpha\alpha}(k'), \quad \frac{1}{2} \langle w_{\alpha}^{\circ 2} \rangle = \int_0^k dk' F_w(k'), \quad (16)$$

which are in reality the cumulative spectral distributions.

C. Boundary Layer

In boundary layers, the parameters (15) are restricted to the components

$$\nabla_3 \bar{u}, \quad \nabla_3 \bar{w}, \quad \text{with} \quad \hat{\underline{w}} = \begin{pmatrix} 0 \\ 0 \\ \hat{w} \end{pmatrix}, \quad (17)$$

so that the energy equations (12) and (13) become reduced into the following particular form:

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u_1^{\circ 2} \rangle = P_{11}^{\circ} + \varphi_{11}^{\circ} - T_{11}^{\circ} \quad (18a)$$

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u_2^{\circ 2} \rangle = \varphi_{22}^{\circ} - T_{22}^{\circ} \quad (18b)$$

$$\frac{1}{2} \frac{\partial}{\partial t} \langle u_3^{\circ 2} \rangle = \varphi_{33}^{\circ} - T_{33}^{\circ} \quad (18c)$$

$$\frac{1}{2} \frac{\partial}{\partial t} \langle w^{\circ 2} \rangle = P_w - T_w \quad (18d)$$

The dissipation functions $\xi_{\alpha\alpha}^{\circ}$, ξ_w° are negligible for large scale turbulence.

D. Kinetic Representation

It can be shown that the hydrodynamical equations of turbulence (1a) and (1b) are simply the zeroth and first moments of the master equation

$$(\partial_t + \underline{v} \cdot \underline{\nabla} - \nu \nabla^2 + \underline{\xi} \cdot \underline{\nabla}) \hat{f}(t, \underline{x}, \underline{v}) = 0, \quad \underline{\nabla} \equiv \partial / \partial \underline{v}, \quad (19)$$

with the equivalence relation

$$\hat{f}(t, \underline{x}, \underline{v}) = \delta[\underline{v} - \hat{\underline{u}}(t, \underline{x})] \quad (20)$$

in the micro-dynamical state. The master equation has the advantage of being homogeneous, and its advective velocity does not cause a nonlinearity since \underline{v} is an independent variable.

Like the hydrodynamical equations, the master equation can be submitted to group-scalings, so that the evolution of $\hat{f}(t, \underline{x}, \underline{v})$ can be shown to be controlled by the transport property.

$$\underline{D}' = \int_0^t d\tau \langle \underline{\xi}'(\underline{t}, \underline{x}) \wedge(t, t-\tau) \underline{\xi}'(\underline{t}) \rangle, \quad (21)$$

called diffusivity, with the evolution operator $\Lambda(t, t-\tau)$. This transport property attains its equilibrium by a relaxation process, that is prescribed by $\Lambda(t, t-\tau)$ and achieved by the diffusion of the trajectory at a diffusivity \mathcal{D} . If the two diffusivities \mathcal{D}' and \mathcal{D}'' are structurally similar, a closure is found.

The analysis of the transport functions (18) will encounter many difficulties from the hydrodynamical approach. It is hoped that the above relaxational closure and the advantages of dealing with a homogeneous master equation of lesser nonlinearity in the kinetic approach will provide with a more powerful tool for the determination of the transport functions.

III. KINETIC FOUNDATION OF THE PRESSURE-STRAIN CORRELATION

Upon scaling the master equation to the macro-group, we have

$$(\partial_t + v_m \nabla_m - \gamma \nabla^2 + A^\circ \xi_m^\circ \partial_m) f^\circ = -\xi_m^\circ \partial_m \bar{f} \quad (22)$$

with the solution

$$f^\circ = -A^\circ \int_0^t d\tau \Lambda(t, t-\tau) \xi_m^\circ(t-\tau) \partial_m \bar{f} \quad (23)$$

By taking the moment and differentiate, we find

$$\begin{aligned} \nabla_\beta u_\alpha^\circ &= -\nabla_\beta \int d v_\alpha \int_0^t d\tau \Lambda(t, t-\tau) \xi_m^\circ(t-\tau) \partial_m \bar{f} \\ &= \nabla_\beta u_\alpha^\circ \Big|_A + \nabla_\beta u_\alpha^\circ \Big|_B \quad (24) \end{aligned}$$

with

$$\nabla_\beta u_\alpha^\circ \Big|_A \approx -\int d v_\alpha \int_0^t d\tau \bar{U}(t, t-\tau) \xi_m^\circ(t-\tau) \partial_m \nabla_\beta \bar{f} \quad (25a)$$

$$\nabla_{\beta} u_{\alpha}^{\circ}|_B = - \int d\nu \, \nu_{\alpha} \int_0^t d\tau [\nabla_{\beta} \bar{U}(t, t-\tau) \xi_m^{\circ}(t-\tau)] \partial_m \bar{f}, \quad (25b)$$

and the divergence-free condition:

$$\nabla \cdot \underline{u}^{\circ} = 0. \quad (26)$$

The differentiation by the operator ∇_{β} has yielded the two contributions (25a) and (25b), and we have approximated the exact evolution operator Λ by the approximate operator \bar{U} , called the average propagator. Now we rewrite these two terms in Fourier space, as follows:

$$\nabla_{\beta} u_{\alpha}^{\circ}|_A \approx \int d\nu \, \nu_{\alpha} \int_0^t d\tau \, \xi_m^{\circ}(k'') e^{i\vec{k}'' \cdot \underline{x}} \langle e^{-i\vec{k}'' \cdot \hat{\underline{l}}(\tau)} \rangle h_{\nu}(\tau) \partial_m \nabla_{\beta} \bar{f} \quad (27a)$$

$$\nabla_{\beta} u_{\alpha}^{\circ}|_B \approx \int d\nu \, \nu_{\alpha} \int_0^t d\tau \int d\vec{k}'' \, \xi_m^{\circ}(k'') i k_{\beta}'' e^{i\vec{k}'' \cdot \underline{x}} \langle e^{-i\vec{k}'' \cdot \hat{\underline{l}}(\tau)} \rangle h_{\nu}(\tau) \partial_m \bar{f}. \quad (27b)$$

Here

$$h_{\nu}(\tau) \equiv e^{-i\vec{k}'' \cdot \underline{v} \tau} \quad (28a)$$

is an orbit function due to the unperturbed streaming, as opposed to the orbit function

$$h_{\ell}(\tau) \equiv \langle e^{-i\vec{k}'' \cdot \hat{\underline{l}}(\tau)} \rangle \quad (28b)$$

due to the perturbed path $\hat{\underline{l}}(\tau)$ by turbulence.

The \underline{v} -integration by parts gives

$$\int d\nu \, \nu_{\alpha} h_{\nu} \partial_m \nabla_{\beta} \bar{f} = i k_{\beta}'' \tau \nabla_{\beta} \bar{u}_{\alpha} \quad (29a)$$

$$\int d\nu \, \nu_{\alpha} h_{\nu} \partial_m \bar{f} = -\delta_{\alpha m} \psi(k'') \quad (29b)$$

Note that the effect of the orbit function! h_{ν} is small as compared to that of h_{ℓ} . The factor $\psi(k'')$ has the role of securing the divergence-free condition.

After the ν -integration, we reduce (27) into:

$$\nabla_{\beta} u_{\alpha}^{\circ}|_A \approx \int_0^t d\tau \tau \int d\mathbf{k}'' i k_{\beta}'' \xi_m^{\circ}(\mathbf{k}'') \langle e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle e^{i\mathbf{k}'' \cdot \mathbf{x}} \nabla_{\beta} \bar{u}_{\alpha} \quad (30a)$$

$$\nabla_{\beta} u_{\alpha}^{\circ}|_B \approx \int_0^t d\tau \int d\mathbf{k}'' i k_{\beta}'' \chi(\mathbf{k}'') \xi_{\alpha}^{\circ}(\mathbf{k}'') \langle e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle e^{i\mathbf{k}'' \cdot \mathbf{x}} \quad (30b)$$

By definition (3a), the pressure

$$p(\mathbf{x}) = \int d\mathbf{k} \ p(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

has a Fourier component

$$\begin{aligned} p(\mathbf{k})/\rho_0 &= \frac{i k_l}{k^2} E_l(\mathbf{k}) \\ &= \frac{i k_l}{k^2} [\xi_l(\mathbf{k}) - E_l(\mathbf{k})] \end{aligned} \quad (31)$$

so that the pressure-strain correlation becomes

$$\frac{1}{\rho_0} \langle p^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle = \frac{1}{\rho_0} \langle p^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle_A + \frac{1}{\rho_0} \langle p^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle_B \quad (32)$$

with

$$\begin{aligned} \frac{1}{\rho_0} \langle p^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle_A &= \int_0^t d\tau \tau \int d\mathbf{k}'' \frac{k_{\beta}'' k_{\alpha}''}{k''^2} \chi \langle E_l^{\circ}(-\mathbf{k}'') \xi_m^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle \nabla_{\beta} \bar{u}_{\alpha} \\ &= D_{lm}^{\circ} \{ \tau k_{\beta}'' k_{\alpha}'' / k''^2 \} \nabla_{\beta} \bar{u}_{\alpha} \approx D^{\circ} \{ \tau \} \nabla_{\beta} \bar{u}_{\alpha} \end{aligned} \quad (33a)$$

$$\begin{aligned} \frac{1}{\rho_0} \langle p^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle_B &= - \int_0^t d\tau \int d\mathbf{k}'' \Delta_{\beta l}(\mathbf{k}'') \chi \langle E_l^{\circ}(-\mathbf{k}'') \xi_{\alpha}^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle \\ &= - D_{l\alpha}^{\circ} \{ \Delta_{\beta l}(\mathbf{k}'') \} \approx - D^{\circ} \{ \Delta_{\beta \alpha}(\mathbf{k}'') \}, \quad \Delta_{\beta \alpha} = k_{\beta}'' k_{\alpha}'' / k''^2 - \frac{1}{3} \delta_{\beta \alpha} \end{aligned} \quad (33b)$$

$$\begin{aligned} D_{lm}^{\circ} &= \int_0^t d\tau \int d\mathbf{k}'' \chi \langle E_l^{\circ}(-\mathbf{k}'') \xi_m^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle \\ &= D_{lm}^{\circ} - D_{lm}^{\circ} \end{aligned} \quad (34a)$$

$$\mathcal{D}_{lm}^{\circ} \equiv \int_0^t d\tau \int d\mathbf{k}'' \chi \langle \xi_l^{\circ}(-\mathbf{k}'') \xi_m^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle \quad (34b)$$

$$\begin{aligned} \mathcal{D}_{lm}^{\circ} &\equiv \int_0^t d\tau \int d\mathbf{k}'' \chi \langle \xi_l^{\circ}(-\mathbf{k}'') \xi_m^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle \\ &= \gamma \int_0^t d\tau \int d\mathbf{k}'' \chi \langle w_l^{\circ}(-\mathbf{k}'') \xi_m^{\circ}(\mathbf{k}'') e^{-i\mathbf{k}'' \cdot \hat{\mathbf{l}}(\tau)} \rangle. \end{aligned} \quad (34c)$$

Here χ is a factor of Fourier truncation, \mathcal{D}_{lm}° is a diffusivity tensor from the auto-correlation of ξ° -field fluctuations, and \mathcal{D}_{lm}° , \mathcal{D}_{lm}° are diffusivity tensors from the cross-correlations. The traces of the tensors do not possess indices. The diffusivities can play the role of integral operators in the integrations with respect to τ and \mathbf{k}'' . Note that (29) and (33) satisfy the divergence-free condition (26), and that

$$\frac{1}{\rho_0} \langle \rho^{\circ} \nabla_{\beta} u_{\alpha}^{\circ} \rangle_A = 0 \quad (35)$$

is a boundary layer turbulence with the parameters (17). It leaves us the correlation (33b). Here the reduction of the diffusivity \mathcal{D}_{lm}° into \mathcal{D}_{lm}° and \mathcal{D}_{lm}° by (34a) has the benefit of confining ourselves to the velocity fluctuations u° , w° , by avoiding the pressure-field fluctuations.

The diffusivity \mathcal{D}° from the auto-correlation has been investigated earlier by Tchen². It is found that for a strongly stable boundary layer, i.e.,

$$|\nabla \bar{u}| \ll N, \quad (36)$$

the effect from \mathcal{D}° is negligible as compared to that from \mathcal{D}° . Here N is the Brunt-Väisälä frequency. The latter diffusivity can be investigated by the same method indicated above. Without going into the details of the calculation which will be reported at a later opportunity, we expect that

$$\varphi_{22}^0 \cong N |\nabla \bar{u}| K^0 \quad (37)$$

The cascade transfer function is

$$T_{22}^0 = K' \langle (\nabla u^0)^2 \rangle \quad (38)$$

from our earlier work. The eddy viscosities are K^0 and K' . Hence the spectral balance (18b) becomes

$$\gamma_t \langle u_2^{02} \rangle = N |\nabla \bar{u}| K^0 - K' \langle (\nabla u^0)^2 \rangle. \quad (39)$$

For a steady state we differentiate (39) with respect to k , and obtain

$$N |\nabla \bar{u}| \dot{K}^0 - K' \frac{d}{dk} \langle (\nabla u^0)^2 \rangle \cong 0 \quad (40a)$$

or

$$N |\nabla \bar{u}| \dot{K}^0 - 2K' k^2 F_{22}(k) = 0, \quad (40b)$$

by writing

$$\langle (\nabla u^0)^2 \rangle = 2 \int_0^k dk' k'^2 F(k') \quad (41)$$

The spectrum is thus found to be

$$F_{22}(k) = \alpha N |\nabla \bar{u}| k^{-3}, \quad (42)$$

where

$$\alpha = \frac{1}{2} k \dot{K}^0 / K' \quad (43)$$

is a factor without dimension. The existence of the spectrum of power law can be seen from the experiments in Fig. 1. This spectrum differs from the power law

$$F_{22}(k) = \alpha N^2 k^{-3} \quad (44)$$

suggested in the literature. The two spectra (42) and (44) are in the ratio

$$|\nabla \bar{u}| / N < 1 \quad (45)$$

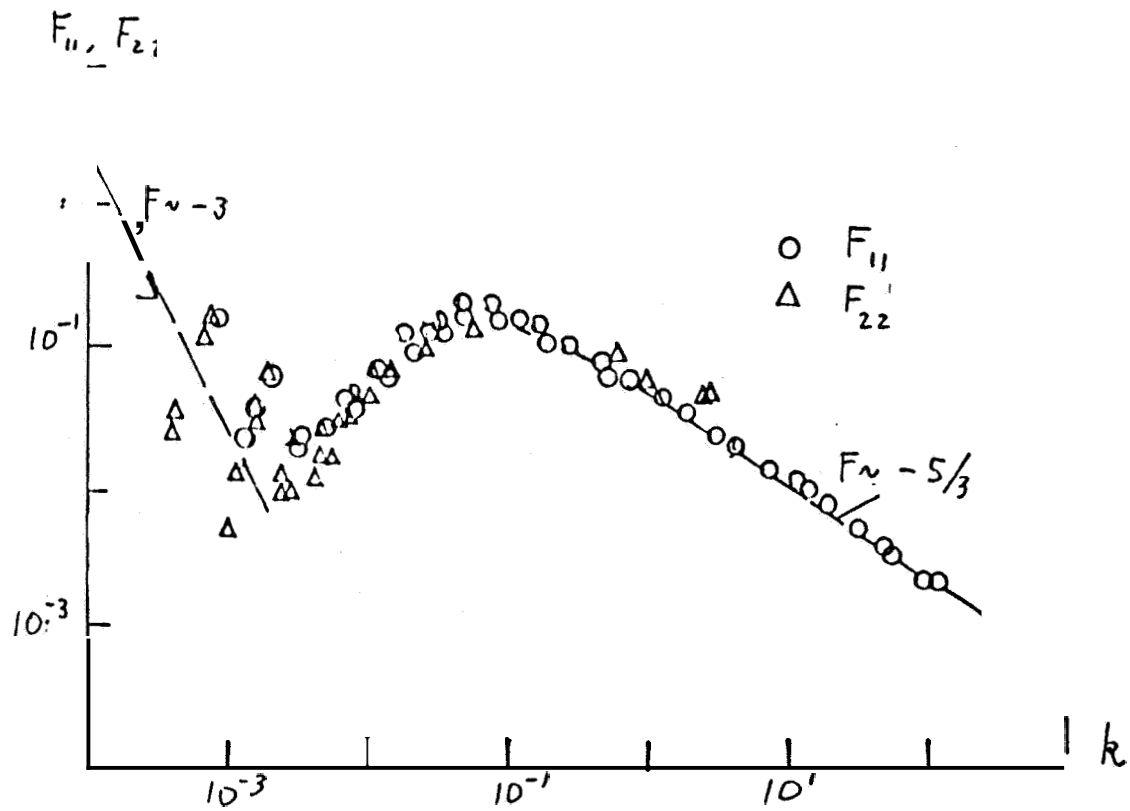


Fig. 1. Normalized spectra $F_{11}(k)$, $F_{22}(k)$ in strongly stable turbulence.

for stable boundary layer. This small ratio is also verified by experiments.

For a very stable boundary layer, where the condition (45) is satisfied, the spectral distribution $\bar{F}_u(k)$ is similarly dominated by the balance between the redistribution and the cascade transfer, so that the power law k^{-3} is again valid. With the strong buoyancy, the spectrum $\bar{F}_{33}(k)$ decays rapidly by going to large k , without giving the opportunity for $\varphi_{33}^0(k)$ to act efficiently.

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SECTION 6

Kinetic Equation of Turbulence

ABSTRACT

The different types of turbulence in applications to the atmospheric and oceanic motions, the propagation of light, the solitary waves, and the plasmas have different governing equations. We shall bring them to a common Liouville form. In this manner, we hope that a single statistical method can be made available to treat the different types of turbulence.

I. FLUID EQUATIONS OF TURBULENCE

Many basic phenomena in space, astrophysics, optics, atmospheric and oceanic sciences are nonlinear and random, i.e. they are in a turbulent state. A proper understanding of these phenomena depends critically on our ability to analyze the turbulent behavior. Although turbulence may appear in different forms and with different basic equations for the description of their microdynamical-states, there exists certain similarity. To exploit the analogy in the attempt of finding a common form of equation of microdynamical state, we first consider the incompressible and homogeneous turbulence, and write the governing Navier-Stokes equations, as follows:

$$\left(\partial_t + \underline{\hat{u}} \cdot \underline{\nabla} - \nu \nabla^2 \right) \underline{\hat{u}} = \underline{\hat{E}}, \quad \underline{\hat{E}} = -\frac{1}{\rho} \underline{\nabla} \hat{p}, \quad \partial_t \equiv \partial / \partial t \quad (1)$$

$$\underline{\nabla} \cdot \underline{\hat{u}} = 0. \quad (2)$$

Here $\underline{\hat{u}}(t, x)$ is the velocity, ρ is the constant density in the incompressible fluid, \hat{p} is the pressure, whose gradient enters into the field $\underline{\hat{E}}$. The variable in the symbol $(\hat{\quad})$ represents the total motion which can be decomposed into a mean and a fluctuation. The kinematic viscosity ν , which is a transport coefficient from the molecular dissipation, is negligibly small compared with its turbulent counter part, and can be omitted in the analysis of the turbulent transport processes. The force of gravity forms a buoyancy, and together with the Coriolis term may be included in the $\underline{\hat{E}}$ field.

By applying the divergence-free condition (2) to the equation of momentum (1), we obtain the following relation between $\underline{\hat{E}}$ and $\underline{\hat{u}}$, in the form

$$\underline{\nabla} \cdot \underline{\hat{E}} = \underline{\nabla} \cdot \underline{\nabla} \cdot \underline{\hat{u}} \underline{\hat{u}} \equiv \hat{\chi}(t, x), \quad (3)$$

called the "equation of state".

The equations of motion of an inviscid compressible fluid are as follows:

$$\partial_t \hat{p} \hat{u}_i + \nabla_j \hat{p} \hat{u}_j \hat{u}_i = \hat{p} \hat{E}_i \quad (4)$$

$$\partial_t \hat{p} + \nabla_j \hat{p} \hat{u}_j = 0. \quad (5)$$

By differentiating (5) with respect to t and eliminating $\partial_t \hat{p} \hat{u}_i$, we obtain the relation

$$\nabla_i \hat{p} \hat{E}_i = -\partial_t^2 \hat{p} + \nabla_i \nabla_j \hat{p} \hat{u}_j \hat{u}_i \equiv \hat{\chi}(t, x). \quad (6)$$

The diffusion of a scalar $\hat{\psi}(t, x)$ with chemical reaction \hat{E} is governed by the following equation

$$(\partial_t + \hat{u} \cdot \nabla - \nu \nabla^2) \hat{\psi} = \hat{E}, \quad \nabla \cdot \hat{u} = 0, \quad (7)$$

where ν is the diffusion coefficient, and \hat{u} is the velocity field.

The same equations (7) are valid for the two-dimensional geostrophic turbulence driven by a random field \hat{E} . The geostrophic turbulence has a vorticity

$$\hat{\psi} \equiv \nabla \times \hat{u},$$

with $\nabla = (\partial/\partial x, \partial/\partial y)$ in two-dimensions.

The equation of propagation of laser light has the form:

$$2ik \frac{\partial \hat{u}}{\partial z} + \nabla^2 \hat{u} + k^2 (\hat{\epsilon} - 1 - \eta |\hat{u}|^2) = 0. \quad (8)$$

Here \hat{u} is the light field propagation in the z -direction with an optical

wavenumber k in a medium of fluctuating dielectric coefficient $\hat{\epsilon}$. The term $-q|\hat{u}|^2$ represents the effect of the strong light intensity on the dielectric medium, causing the self-focusing of light.

The equation (8) is called the "nonlinear Schrödinger equation"^{1,2}. It also applies to the solitons in plasmas, the atmosphere and the oceans. Here \hat{u} is the envelop of the waves.

The linearized form

$$zik \frac{\partial \hat{u}}{\partial z} + \nabla_{\perp}^2 \hat{u} + k^2(\hat{\epsilon} - 1)\hat{u} = 0, \quad \text{with } \nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (9)$$

is called the parabolic equation of propagation.^{3,4}

Although the equation (9) is linear in \hat{u} , the product of two random functions $\hat{\epsilon}\hat{u}$ makes the problem statistically nonlinear. In the form (9), the propagation of light resembles the diffusion (7) of a scalar.

Recently a great deal of effort on the nonlinear wave phenomena has been made by using the Korteweg-de Vries (KdV) equation:⁵

$$(\partial_t + \hat{L})\hat{u} = 0, \quad \text{with } \hat{L} \equiv \hat{u} \frac{\partial}{\partial x} + \lambda \frac{\partial^3}{\partial x^3}, \quad (10)$$

where λ is the coefficient of dispersion. It has been shown that this adequately describes the long-time asymptotic behavior of wave motions in nonlinear dispersive systems, and that this asymptotic form can be derived either from the nonlinear Schrödinger equation (8) or from the perturbations of the equations of wave motions.

II. MASTER EQUATION FOR THE DESCRIPTION OF THE MICRODYNAMICAL STATE OF TURBULENCE

Although the fluid turbulence obeys different forms of differential equations (1) - (7) for the description of the microdynamical state, their kinetic representation can be cast into a unified **master** equation, or Klimontovich equation:⁶

$$(\partial_t + \hat{L}) \hat{f}(t, \underline{x}, \underline{v}) = 0. \quad (11)$$

Here $\hat{f}(t, \underline{x}, \underline{v})$ is the distribution of the random variable \underline{x} at time t and position \underline{x} , and $\hat{L}(t, \underline{x}, \underline{v})$ is a differential operator which takes varied forms depending on the types of turbulence in consideration. These forms are found in the following.

(i) For the inviscid incompressible turbulence as governed by (1) - (3), we have the differential operator

$$\hat{L} \equiv \underline{v} \cdot \underline{\nabla} + \underline{\hat{E}} \cdot \underline{\partial}, \quad \underline{\partial} \equiv \partial / \partial \underline{v}, \quad (12)$$

the equation of state

$$\underline{\nabla} \cdot \underline{\hat{E}} = \int d\underline{v} \, n(\underline{x}, \underline{v}) \hat{f}(t, \underline{x}, \underline{v}), \quad (13)$$

and the source

$$n(\underline{x}, \underline{v}) = \underline{\nabla} \underline{\nabla} : \underline{v} \underline{v}. \quad (14)$$

The limits of integration are understood to extend to the whole available domain, unless otherwise specified. The equivalence between the two representations, i.e. the fluid representation (1) - (3) on the one hand and the kinetic representation (11) on the other, can be obtained by writing

$$\hat{f}(t, \underline{x}, \underline{v}) = \delta[\underline{v} - \underline{\hat{u}}(t, \underline{x})]. \quad (15)$$

With this equivalence, it is not difficult to transform the equation of state from the kinetic form (13) into the hydrodynamic form (3), and the master equation (11) into the hydrodynamic system (1) and (2) by means of moments.

(ii) For the compressible turbulence, as governed by (4) - (6), we have the same differential operator (12) and the same equation of state (13).

The equivalence between the two representations is obtained by writing the distribution as

$$\hat{f}(t, \underline{x}, \underline{v}) = \hat{\rho} \delta[\underline{v} - \hat{\underline{u}}(t, \underline{x})], \quad (16)$$

the equation of state as

$$\underline{\nabla} \cdot \hat{\underline{E}} = \int d\underline{v} \, \rho(t, \underline{x}, \underline{v}) \hat{f}(t, \underline{x}, \underline{v}) \quad (17a)$$

and the source as

$$\rho(t, \underline{x}, \underline{v}) = -\partial_t^2 + \underline{\nabla} \cdot \underline{\nabla} : \underline{v} \underline{v} \quad (17b)$$

By the equivalence relation (16), it is not difficult to transform the kinetic representation with formulas (11) and (16) into the fluid representation with formulas (4) - (6).

(iii) For the diffusion and the two-dimensional geostrophic turbulence, as governed by (7), the Liouville equation (11) remains valid, with a differential operator

$$\hat{L} = \hat{\underline{u}} \cdot \underline{\nabla} - \nu \nabla^2, \quad (18)$$

and an equivalence condition

$$\hat{f}(t, \underline{x}, \underline{v}) = \delta[\underline{v} - \hat{\underline{\psi}}(t, \underline{x})]. \quad (19)$$

(iv) For the light propagation, we can introduce

$$t = \tilde{t}/c, \quad \nu = c/2k. \quad (20)$$

and transform the Schrödinger equations (8) and (9) into

$$(\partial_t + \hat{L})\hat{u} = 0, \quad \hat{L} = -i\nu[\nabla^2 + \hat{\phi}(\hat{\epsilon}, \hat{u})] . \quad (21)$$

with

$$\hat{\phi} \equiv -i\nu k^2(\hat{\epsilon} - 1 - q|\hat{u}|^2) \quad (22a)$$

in nonlinear propagation, and

$$\hat{\phi} \equiv -i\nu k^2(\hat{\epsilon} - 1) \quad (22b)$$

in linear propagation.

Note that (21) is already in a form analogous to the **master** equation so that further transformation into the kinetic representation becomes unnecessary. The same argument holds for the KdV equation (10).

(v) The plasma turbulence is governed by (11), now called the Vlasov equation, with the differential operator (12). The \hat{E} - field represents the electrostatic field with unit charge and mass. The equation of state in the form (13) remains valid, with the source

$$\rho(x, \nu) = \omega_p^2 \quad (23)$$

where ω_p is the constant plasma frequency. The distribution function has the normalization condition

$$\int d\nu \hat{f} = \hat{n} \quad , \quad (24)$$

where $\hat{n}(t, x)$ is the number-density,

TYPES OF TURBULENCE	LIIOUVILLE EQUATION	DISTRIBUTION	DIFFERENTIAL OPERATOR	SOURCE
(a) incompressible turbulence	$(\partial_t \hat{L}) \hat{f} = 0$	$\hat{f} = \delta(\vec{v} - \hat{u})$	$\hat{L} = \vec{v} \cdot \nabla + \hat{E} \cdot \partial$	$\mathcal{L} = \nabla \cdot \vec{v} \vec{v}$
(b) compressible turbulence	"	$\hat{f} = \hat{\rho} \delta(\vec{v} - \hat{u})$	"	$\mathcal{L} = -\partial_t^2 + \nabla \cdot \vec{v} \vec{v}$
(c) plasma turbulence	"	$\int d\vec{v} \hat{f} = \hat{n}$	"	$\mathcal{L} = \omega_p^2$
(d) diffusion in turbulence	"	$\hat{f} = \delta(\vec{v} - \psi)$	$\hat{L} = \hat{u} \cdot \nabla + \hat{E} \cdot \partial$	
(e) geostrophic turbulence	"	"	"	
(f) light propagation in turbulence	$(\partial_t \hat{L}) \hat{u} = 0$		$\hat{L} = -i\lambda(\nabla^2 + \hat{\varphi})$	
(g) solitons in turbulence			$\hat{L} = \hat{u} \frac{\partial}{\partial x} + \lambda \frac{\partial^3}{\partial x^3}$	

Table 1. Different types of turbulence in the Liouville form.

In Table 1, we have listed the different types of turbulence. Their governing equations are written in the **master** form with appropriate differential operators. The self-consistent field is related to the distribution function by the equation of state in the form (13) through a source λ . This reduction of several types of turbulence into a unified **master** equation suggests that they can all be treated by a single statistical method.

The description of the microdynamical state of turbulence by an equation of the **master** form presents the advantages that the equation is homogeneous, and that the replacement of the velocity function $\hat{u}(t, \underline{x})$ by the independent variable \underline{v} prevents any terms connected with the velocity from becoming nonlinear, as found in the differential operators \hat{L} and the sources (a) - (c).

The only surviving nonlinear term $\hat{\underline{E}} \cdot \nabla \hat{f}$ arises from the $\hat{\underline{E}}$ -field. Fortunately this does not require our immediate attention, since it can be provisionally considered as a given random force until after the closure. Thereafter, $\hat{\underline{E}}$ is determined by the equation of state with the proper source.

This nonlinear term describes the wave-particle interaction explicitly, and therefore will enable the derivation of the nonlinear Landau damping in fluids in a direct manner. The fluid representation may describe such an interaction only indirectly through a series of moments.

3. Kinetic Hierarchy of Turbulence

In the following lines we shall be concerned with the incompressible Navier-Stokes turbulence. The microdynamical state of turbulence is described in the kinetic representation by the **master** equation

$$(\partial_t + \hat{L}) \hat{f} = 0, \quad (25)$$

with a differential operator

$$\hat{L} = \underline{v} \cdot \underline{\nabla} - \nu \nabla^2 + \hat{\underline{E}} \cdot \underline{\partial} \quad (26)$$

and a self-consistent field

$$\hat{\underline{E}} = -\frac{1}{\rho} \nabla \hat{p}, \quad (27)$$

so that the pressure \hat{p} satisfies the Poisson equation

$$\begin{aligned} \nabla^2 \hat{p} &= -\rho \nabla \cdot \hat{\underline{E}} \\ &= -\rho \int d\underline{v} \, \underline{u}(\underline{x}, \underline{v}) \cdot \hat{f}(\underline{t}, \underline{x}, \underline{v}), \end{aligned} \quad (28)$$

by (13). The solution can be written in the form:

$$\hat{p}(\underline{t}, \underline{x}) = \rho \iint d\underline{x}' d\underline{v}' \frac{1}{4\pi |\underline{x} - \underline{x}'|} \underline{u}(\underline{x}', \underline{v}') \cdot \hat{f}(\underline{t}, \underline{x}', \underline{v}'). \quad (29)$$

We calculate

$$\hat{\underline{E}}(\underline{t}, \underline{x}) = -\nabla \iint d\underline{x}' d\underline{v}' \frac{1}{4\pi |\underline{x} - \underline{x}'|} \underline{u}(\underline{x}', \underline{v}') \cdot \hat{f}(\underline{t}, \underline{x}', \underline{v}') \quad (30)$$

by differentiating (29), and obtain

$$\hat{\underline{E}} \cdot \underline{\partial} \hat{f} = \rho(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \left\{ \hat{f}(\underline{t}, \underline{x}, \underline{v}) \dot{\hat{f}}(\underline{t}, \underline{x}', \underline{v}') \right\}, \quad (31)$$

by introducing the integral operator

$$g(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \equiv - \partial \cdot \int d\underline{x}' d\underline{v}' \nabla \left(\frac{1}{4\pi |\underline{x} - \underline{x}'|} \right) h(\underline{x}', \underline{v}'). \quad (32)$$

The term (31) yields a second order nonlinearity in the master equation.

The master equation, which describes the microdynamical state of turbulence, contains fluctuations with all minute details which are unnecessary, if not impossible, in a statistical study. A coarse-graining procedure is the "global ensemble average" with the operator

$$\bar{A} \equiv \langle \rangle.$$

Such an average is deterministic and may vary with t_x or $t_{x,v}$. The deviation from this average is obtained by the operator of fluctuation

$$\tilde{A} = 1 - \bar{A},$$

where "1" is the unit operator.

By applying the operators \bar{A}, \tilde{A} to (25), we obtain:

$$(\partial_t + \bar{L}) \bar{f} = - \bar{A} \tilde{L} \tilde{f} \equiv \bar{C}, \quad (33)$$

and

$$(\partial_t + \hat{L}) \tilde{f} = - \tilde{L} \bar{f} - \bar{C}. \quad (34)$$

or

$$(\partial_t + \tilde{A} \hat{L}) \tilde{f} = - \tilde{L} \bar{f}. \quad (35)$$

The operators operate on the functions which follow.

The equation (33) describes the evolution of the distribution $\bar{f}(t, \underline{x}, \underline{v})$ in a turbulent medium which presents a turbulent collision $\bar{C}(t, \underline{x}, \underline{v})$

representing the statistical effects of the fluctuations $\tilde{L}\tilde{f}$ upon \bar{f} . Note that the collision, as defined by

$$\begin{aligned}\bar{C} &\equiv -\bar{A}\tilde{L}\tilde{f} \\ &= -\tilde{\partial} \cdot \bar{A} \tilde{E}\tilde{f},\end{aligned}\quad (36)$$

is the derivative of the Reynolds stress

$$-\bar{A} \tilde{E}\tilde{f} \quad (37)$$

in the phase space. In this sense the equation (33) can be called the Reynolds equation in the phase space, and the equation of evolution of the fluctuation, (34) or (35), is the Friedmann equation, as obtained by the Reynolds decomposition

$$f = \bar{A} + \tilde{A}. \quad (38)$$

By this decomposition, we can separate (32) into two components, as follows:

$$-\tilde{E} \cdot \tilde{\partial} \bar{f} = g(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \{ \bar{f}(t, \underline{x}, \underline{v}) \bar{f}(t, \underline{x}', \underline{v}') \} \quad (39a)$$

$$-\bar{A} \tilde{E} \cdot \tilde{\partial} \tilde{f} = g(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \{ \bar{f}_{12}(t, \underline{x}, \underline{v}; t, \underline{x}', \underline{v}') \} \quad (39b)$$

where

$$\bar{f}_{12}(t, \underline{x}, \underline{v}; t, \underline{x}', \underline{v}') \equiv \bar{A} \tilde{f}(t, \underline{x}, \underline{v}) \tilde{f}(t, \underline{x}', \underline{v}') \quad (40)$$

is the pair distribution function. Upon substituting (39) into the ensemble average of (32) and subsequently into (33), we get the following equation of evolution of \bar{f} :

$$\left(\partial_t + \underline{v} \cdot \underline{\nabla} - \nu \nabla^2 \right) \bar{f}(t, \underline{x}, \underline{v}) = g(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \left\{ \bar{f}(t, \underline{x}, \underline{v}) \bar{f}(t, \underline{x}', \underline{v}') \right\} + \bar{C} , \quad (41)$$

with

$$\bar{C}(t, \underline{x}, \underline{v}) \equiv g(\underline{x}, \underline{v} | \underline{x}', \underline{v}') \left\{ \bar{f}_{12}(t, \underline{x}, \underline{v}; t, \underline{x}', \underline{v}') \right\} . \quad (42)$$

The dependance of the collision upon the pair distribution function \bar{f}_{12} is expected. If we consider a source of the type (23), we reduce (41) to the BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon) hierarchy of plasmas. Our general form (41) applies to other types of turbulence too.⁷⁻¹²

The equation of evolution of \bar{f} in (41) will be called the kinetic equation of turbulence. It remains to determine the collision as an explicit function of \bar{f} , i.e. in the form:

$$\bar{C} = \bar{C} \{ \bar{f} \} . \quad (43)$$

where \bar{C} is a collision operator.

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16. ABSTRACT Two-dimensional geostrophic turbulence driven by a random force is investigated. Based on the Liouville equation, which simulates the primitive hydrodynamical equations, a group-kinetic theory of turbulence is developed and the kinetic equation of the scaled singlet distribution is derived. This distribution will suffice for the investigation of the spectrum of turbulence. The collision integral has a memory and describes the pair interaction and its enhancement by the multiple interaction. The kinetic equation is transformed into an equation of spectral balance in the equilibrium and non-equilibrium states. The propagator formalism is summarized in a self-consistent way for the asymptotic quasi-linear equation. Comparison is made between the propagators and the Green's functions in the case of the non-asymptotic quasi-linear equation to prove the equivalence of both kinds of approximations used to describe perturbed trajectories of plasma turbulence. The microdynamical state of fluid turbulence is described by a hydrodynamical system and transformed into a master equation analogous to the Vlasov equation for plasma turbulence. When the total distribution function is decomposed into a mean value and a fluctuation, the evolution of the mean distribution satisfies a transport equation, i.e., kinetic equation, and contains a turbulent collision that represents the statistical effect of the turbulent fluctuation, while the evolution of the fluctuation will form a transport equation for the collision. The hydrodynamical equations of turbulence are transformed into a master equation for the velocity distribution function. A group-scaling is introduced for the closure. The spectral balance for the velocity fluctuations of individual components shows that the scaled pressure-strain correlation and the cascade transfer are two transport functions that play the most important roles.					
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